

# Traces on Module Categories over Fusion Categories

Gregor Schaumann\*

Department Mathematik  
Friedrich-Alexander Universität Erlangen-Nürnberg  
Cauerstraße 11  
91058 Erlangen  
Germany

June 26, 2012

## Abstract

We consider traces on module categories over pivotal fusion categories which are compatible with the module structure. It is shown that such module traces characterise the Morita classes of special haploid Frobenius algebras. Moreover, they are unique up to a scale factor and they equip the dual category with a pivotal structure. This implies that for each pivotal structure on a fusion category over  $\mathbb{C}$  there exists a conjugate pivotal structure defined by the canonical module trace.

## 1 Introduction

Fusion categories exhibit a rich mathematical structure, see for example [7, 16]. They have important applications in 3-dimensional topological field theory [1, 18], in particular in the study of invariants of 3-manifolds [2, 20], and in rational conformal field theory, see [15], [9] and subsequent work. The construction in conformal field theory initiated in [9] requires as starting point a special haploid Frobenius object in a modular fusion category, but it depends only on the Morita class of that algebra. It is known [17] that Morita classes of algebras in fusion categories are described by equivalence classes of module categories.

In this article we provide a description of the Morita classes of special haploid Frobenius algebras in pivotal fusion categories over  $\mathbb{C}$  in terms of module categories with module traces. A module trace is a trace on a module category, i.e. a collection of symmetric and non-degenerate linear maps from the endomorphism spaces of objects to  $\mathbb{C}$ , that is compatible with the module structure. As a main result we prove the following:

**Theorem** *Let  $\mathcal{C}$  be a pivotal fusion category. The following structures are equivalent:*

- i) *An indecomposable module category  ${}_e\mathcal{M}$  with module trace.*
- ii) *An indecomposable module category  ${}_e\mathcal{M}$  together with a  $\mathcal{C}$ -balanced natural isomorphism between  $\mathrm{Hom}(n, m)$  and the dual space of  $\mathrm{Hom}(m, n)$ , for each pair of objects  $m, n \in \mathcal{M}$ .*
- iii) *A Morita class of a special haploid Frobenius algebra in  $\mathcal{C}$ .*

---

\*email: gregor.schaumann@math.uni-erlangen.de

The equivalence of *i)* and *ii)* implies that module traces on indecomposable module categories are unique up to a constant factor and equip the dual fusion category with a pivotal structure. When applied to the particular case of  $\mathcal{C}$  considered as a left module category over itself, we obtain the following result.

**Theorem** *For each pivotal structure  $a$  on a fusion category  $\mathcal{C}$  over  $\mathbb{C}$  there exists a conjugate pivotal structure  $\bar{a}$  such that the right dimensions of objects with respect to  $\bar{a}$  are complex conjugate to the right dimensions with respect to  $a$ .*

We show how this result is related to the existence of a natural monoidal isomorphism of the identity and the quadruple dual functor for fusion categories from [7].

We give an explicit description of module traces in terms of a matrix equation that provides a reduction of the problem of solving a quadratic equation for algebras (the Frobenius property) to a linear equation for the module category. This implies in particular that the quantum dimensions of special haploid Frobenius algebras in pivotal fusion categories are positive real numbers and shows that all module categories over pseudo-unitary fusion categories admit a module trace. We extend the graphical calculus for tensor categories to module categories and give a graphical description of the Frobenius algebra obtained from a module category with module trace.

In [5] it is shown that indecomposable module categories over a fusion category  $\mathcal{C}$  are classified by Lagrangian algebras in the Drinfeld center  $\mathcal{Z}(\mathcal{C})$ . It remains to interpret our results in terms of this classification.

A possible application of our results is to modify the construction in [9] in such a way that it depends only on a module category with module traces over a modular fusion category and involves no further choices. In such a construction it should be possible to incorporate module functors and module natural transformations as well and interpret them in physical terms, see [19, Sec. 3], [14] for a possible interpretation.

The paper is structured as follows. In Section 2 we summarise the relevant background about fusion categories, algebra objects and module categories. In Section 3 we first develop a graphical notation for module categories which gives rise to a diagrammatic description of the algebra structure of inner hom objects. Next we introduce module traces and demonstrate in examples that the existence of a module trace for a given module category depends on the choice of pivotal structure for the fusion category. In Section 4 we give a description of module traces in terms of  $\mathcal{C}$ -balanced natural isomorphisms and prove that module traces on indecomposable module categories are unique up to scaling. This description of module traces yields a module natural isomorphism between a module functor and its double adjoint functor. In the application to a pivotal fusion category as a module category over itself, this leads to the existence of conjugate pivotal structures for pivotal fusion categories. We provide a graphical derivation of a monoidal natural isomorphism of the identity functor to the quadruple dual functor for fusion categories and show that this yields an alternative definition of the conjugate pivotal structure. In Section 5 we demonstrate that the existence of a module trace can be reduced to a matrix equation and discuss the example of pseudo-unitary fusion categories. As a consequence of these results we obtain a new criterion to decide whether a pivotal structure is spherical in terms of module categories. In Section 6 we prove that module traces characterise equivalence classes of special haploid Frobenius algebras.

## 2 Preliminaries

### 2.1 Fusion Categories and Algebra Objects

In this section we summarise the relevant background and fix our notation. All categories are assumed to be abelian and moreover locally finite over  $\mathbb{C}$ , i.e. the isomorphism classes

of objects form a set, all  $\mathbf{Hom}$ -spaces are finite dimensional and every object has finite length. All functors and natural transformations are assumed to be additive.

**Definition 2.1** ([6]) *A tensor category  $\mathcal{C}$  is a monoidal category with rigidity and simple unit  $1 \in \mathcal{C}$  such that the monoidal structure is bilinear on morphisms. A finite tensor category is a tensor category with finitely many simple objects up to isomorphism. A fusion category is a semisimple finite tensor category.*

Without loss of generality we will work with strict monoidal categories (see e.g. [1]). Rigidity means that each object  $c \in \mathcal{C}$  has a right dual  $c^*$  with duality morphisms  $\mathbf{ev}_c : c^* \otimes c \rightarrow 1$ ,  $\mathbf{coev}_c : 1 \rightarrow c \otimes c^*$  and a left dual  ${}^*c$  with  $\mathbf{ev}'_c : c \otimes {}^*c \rightarrow 1$  and  $\mathbf{coev}'_c : 1 \rightarrow {}^*c \otimes c$ , such that the rigidity axioms are satisfied, see Appendix A, equation (A.7). Right and left duals are unique up to a unique isomorphism. In a rigid tensor category there is a canonical natural isomorphism  $c \simeq {}^*(c^*) \simeq ({}^*c)^*$  for each object  $c \in \mathcal{C}$  and we will therefore identify these objects in the sequel.

The functor  $(.)^{**}$  has a canonical structure of a tensor functor. A pivotal structure for  $\mathcal{C}$  is a monoidal natural isomorphism  $a : \mathbf{id}_{\mathcal{C}} \rightarrow (.)^{**}$ . In particular, a pivotal structure allows one to define the right trace of a morphism  $f \in \mathbf{End}(c)$  as

$$\mathrm{tr}_c^R(f) = \mathbf{ev}_c \circ (a_{*c} \otimes f) \circ \mathbf{coev}'_c \in \mathbf{End}(1) \simeq \mathbb{C} \quad (2.1)$$

and for each object  $c$  the quantum dimension  $\mathrm{tr}_c^R(\mathbf{id}_c) = \dim^{\mathcal{C}}(c)$ . The left trace of a morphism is defined analogously and a pivotal structure is called spherical if the left traces and right traces agree for all morphisms. Throughout this paper  $\mathcal{C}$  denotes a pivotal fusion category unless stated otherwise. We use the well-established graphical calculus for tensor categories, see Appendix A for relevant definitions and conventions.

## Algebra Objects

**Definition 2.2** *An algebra (object) in a tensor category  $\mathcal{C}$  is an object  $A \in \mathcal{C}$  together with a multiplication morphism  $\mu : A \otimes A \rightarrow A$ , and a unit morphism  $\eta : 1 \rightarrow A$ , represented by the diagrams*

$$\mu \hat{=} \begin{array}{c} \cup \\ | \end{array}, \quad \eta \hat{=} \begin{array}{c} | \\ \bullet \end{array}, \quad (2.2)$$

*such that the associativity and unit constraints hold:*

$$\begin{array}{c} \cup \\ | \end{array} = \begin{array}{c} \cup \\ | \end{array}, \quad \begin{array}{c} \cup \\ | \end{array} \bullet = \begin{array}{c} \cup \\ | \end{array} \bullet = \begin{array}{c} | \end{array}. \quad (2.3)$$

*An algebra  $A$  in  $\mathcal{C}$  is called haploid if  $\mathbf{Hom}_{\mathcal{C}}(1, A) \simeq \mathbb{C}$  as a vector space.*

There is the obvious definition of morphisms of algebras. An algebra is called indecomposable if it is not isomorphic to a direct sum of two non-trivial algebras. As we will always work with just one algebra at a time, we omit the labels on the lines representing the algebra object. Given an algebra in  $\mathcal{C}$ , we can consider modules over this algebra in  $\mathcal{C}$ .

**Definition 2.3** *A right module over an algebra  $A$  in a tensor category  $\mathcal{C}$  is an object  $M \in \mathcal{C}$  together with an action morphism*

$$\rho : M \otimes A \rightarrow M \hat{=} \begin{array}{c} | \\ | \end{array} \begin{array}{c} \cup \\ | \end{array}, \quad (2.4)$$

such that the following equations hold:

$$\begin{array}{c} \text{M} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{M} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array}, \quad \begin{array}{c} \text{M} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{M} \\ \text{---} \end{array}. \quad (2.5)$$

An intertwiner between two right modules  $(M, \rho)$  and  $(N, \chi)$  over  $A$  is a morphism  $\phi : M \rightarrow N$  in  $\mathcal{C}$  which satisfies

$$\begin{array}{c} \text{M} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{M} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array}. \quad (2.6)$$

There are analogous definitions for left modules. The subspace of  $\text{Hom}_{\mathcal{C}}(M, N)$  consisting of the intertwiners is denoted by  $\text{Hom}_A(M, N)$ .

It is clear (see e.g. [17]) that for an algebra  $A$ , a right module  $(M, \rho)$  over  $A$  and an object  $c \in \mathcal{C}$ , the object  $c \otimes M$  is also a right module over  $A$  with action morphism

$$\text{id}_c \otimes \rho : c \otimes M \otimes A \rightarrow c \otimes M, \quad (2.7)$$

and that each morphism  $\phi : c \rightarrow d$  in  $\mathcal{C}$  yields an intertwiner  $\phi \otimes \text{id}_M : c \otimes M \rightarrow d \otimes M$ .

**Definition 2.4** ([10]) An coalgebra (object) in a tensor category  $\mathcal{C}$  is an object  $C \in \mathcal{C}$  together with a comultiplication morphism

$$\Delta : C \rightarrow C \otimes C \cong \begin{array}{c} \text{---} \\ \text{---} \end{array}, \quad (2.8)$$

and a counit morphism

$$\epsilon : C \rightarrow 1 \cong \begin{array}{c} \text{---} \\ \text{---} \end{array}, \quad (2.9)$$

such that the coassociativity and counit constraints hold:

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}, \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}. \quad (2.10)$$

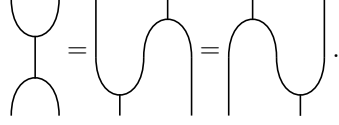
**Definition 2.5** ([5, 10]) Let  $\mathcal{C}$  be a tensor category.

- i) A separable algebra  $A \in \mathcal{C}$  is an algebra  $(A, \mu, \eta)$  for which there exists a morphism  $\Delta : A \rightarrow A \otimes A$  with  $\mu \circ \Delta = \text{id}_A$  and

$$\Delta \circ \mu = (\mu \otimes \text{id}_A) \circ (\text{id}_A \otimes \Delta) = (\text{id}_A \otimes \mu) \circ (\Delta \otimes \text{id}_A). \quad (2.11)$$

- ii) A Frobenius algebra in  $\mathcal{C}$  is an algebra  $(A, \mu, \eta)$  that is also a coalgebra with structures  $\epsilon : A \rightarrow 1$  and  $\Delta : A \rightarrow A \otimes A$ , such that (2.11) is satisfied.

In graphical notation relation (2.11) reads :

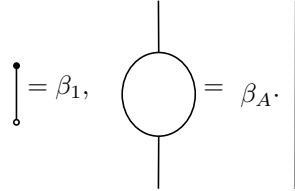

(2.12)

**Lemma 2.6** ( [5], **Prop. 2.7**) *Consider an algebra  $(A, \mu, \eta)$  in a fusion category  $\mathcal{C}$ . Then the category  $\text{Mod}_{\mathcal{C}}(A)$  is semisimple if and only if  $A$  is separable.*

The following Frobenius algebras are particularly important in applications to conformal field theory [9].

**Definition 2.7** ( [10]) *A Frobenius algebra  $A$  in  $\mathcal{C}$  is called*

*i) special if there exist  $\beta_1, \beta_A \in \mathbb{C}^\times$  such that*


(2.13)

*ii) symmetric if*

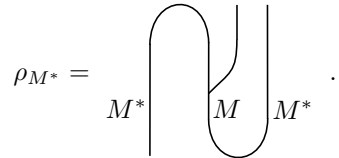

(2.14)

Condition *ii)* can be extended to any algebra  $A$  with a morphism  $\epsilon \in \text{Hom}_{\mathcal{C}}(A, 1)$ .

**Lemma 2.8** ( [8]) *Let  $A$  be a special symmetric Frobenius algebra in  $\mathcal{C}$ . Then  $\dim^{\mathcal{C}}(A) = \beta_1 \beta_A \neq 0$ . We can normalise  $\epsilon$  and  $\Delta$  such that  $\beta_1 = \dim^{\mathcal{C}}(A)$  and  $\beta_A = 1$ .*

**Lemma 2.9** ( [9]) *If an algebra  $A$  is haploid and has dimension  $\dim^{\mathcal{C}}(A) \neq 0$ <sup>1</sup>, then it is symmetric for any choice of  $\epsilon \in \text{Hom}_{\mathcal{C}}(A, 1)$ .*

Let  $\mathcal{C}$  be a pivotal fusion category. With the pivotal structure we will identify left and right dual objects in the remainder of this section. The dual  $M^*$  of a right  $A$ -module  $(M, \rho)$  inherits a canonical left  $A$ -module structure


(2.15)

For a right  $A$ -module  $(M, \rho^M)$  and a left  $A$ -module  $(X, \rho^X)$ , is a notion of the tensor product  $M \otimes_A X$  over  $A$ , see e.g. [10].  $M \otimes_A X$  is an object in  $\mathcal{C}$  that is defined as the cokernel of the map  $(\rho^M \otimes \text{id}_X) - (\text{id}_M \otimes \rho^X) : M \otimes A \otimes X \rightarrow M \otimes X$ . When  $A$  is

---

<sup>1</sup> In the proof [9, Cor. 3.10] the assumption  $\dim^{\mathcal{C}}(A) \neq 0$  is implicitly present. We thank I. Runkel for this information.

a normalised special Frobenius algebra,  $M \otimes_A X$  is equal to the image of the following projector  $P : M \otimes X \rightarrow M \otimes X$ :

$$P = M \left( \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \text{---} \end{array} \right) X . \quad (2.16)$$

**Proposition 2.10** *Let  $A$  be a special haploid Frobenius algebra in a pivotal fusion category  $\mathcal{C}$ . There is a natural isomorphism  $\text{Hom}_A(M, N) \simeq \text{Hom}_{\mathcal{C}}(1, N \otimes_A M^*)$  for  $M, N \in \text{Mod}_{\mathcal{C}}(A)$ .*

*Proof:* This follows from the fact that the inner hom object of  $M$  and  $N$  is  $N \otimes_A M^*$ , see [6].  $\square$

## 2.2 Module Categories

In this subsection we summarise the main definitions and results concerning module categories, see [6, 17] for more details. The following definition is a restriction of the definition in [17] to semisimple categories.

**Definition 2.11** *A left  $\mathcal{C}$ -module category  $\mathcal{M}$  is a semisimple  $\mathbb{C}$ -linear abelian category  $\mathcal{M}$ , together with a bifunctor  $\triangleright : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$  and natural isomorphisms*

$$\omega_{c,d,m} : (c \otimes d) \triangleright m \rightarrow c \triangleright (d \triangleright m), \quad l_M : 1 \triangleright m \rightarrow m, \quad (2.17)$$

for all  $c, d \in \mathcal{C}$ ,  $m \in \mathcal{M}$ , such that the module constraints are fulfilled: The diagrams

$$\begin{array}{ccc} & ((c \otimes d) \otimes e) \triangleright m & \\ \swarrow = & & \searrow \omega_{c \otimes d, e, m} \\ (c \otimes (d \otimes e)) \triangleright m & & (c \otimes d) \triangleright (e \triangleright m) \\ \downarrow \omega_{c, d \otimes e, m} & & \downarrow \omega_{c, d, e \triangleright m} \\ c \triangleright ((d \otimes e) \triangleright m) & \xrightarrow{\text{id}_c \triangleright \omega_{d, e, m}} & c \triangleright (d \triangleright (e \triangleright m)), \end{array} \quad (2.18)$$

and

$$\begin{array}{ccc} (c \otimes 1) \triangleright m & \xrightarrow{\omega_{c, 1, m}} & c \triangleright (1 \triangleright m) \\ & \searrow \text{id}_c \triangleright m & \swarrow \text{id}_c \triangleright l_m \\ & c \triangleright m & \end{array} \quad (2.19)$$

commute for all objects  $c, d, e \in \mathcal{C}$  and  $m \in \mathcal{M}$ . To emphasise that  $\mathcal{M}$  is a left  $\mathcal{C}$ -module category we denote it  ${}_c\mathcal{M}$ . There is an analogous definition of a right  $\mathcal{C}$ -module category  $\mathcal{M}_e$  with an bifunctor  $\triangleleft : \mathcal{M}_e \otimes \mathcal{C} \rightarrow \mathcal{M}_e$  satisfying analogous constraints.

For a left  $\mathcal{C}$ -module category  ${}_c\mathcal{M}$ , the opposite category  $\mathcal{M}^{\text{op}}$  is a right  $\mathcal{C}$ -module category  $\mathcal{M}_e^{\text{op}}$  with action

$$m \triangleleft^{\text{op}} c = c^* \triangleright m. \quad (2.20)$$

**Definition 2.12** ([17]) *i) Let  ${}_e\mathcal{M}$  and  ${}_e\mathcal{N}$  be  $\mathcal{C}$ -module categories. A  $\mathcal{C}$ -module functor  $F : {}_e\mathcal{M} \rightarrow {}_e\mathcal{N}$  is a functor  $F$  together with natural isomorphisms  $f_{c,m} : F(c \triangleright m) \rightarrow c \triangleright F(m)$ , such that the usual pentagon and triangle diagrams commute, see [17]. We sometimes write  $(F, f)$  for a module functor and call  $f$  a left module constraint for  $F$ . Module functors between right  $\mathcal{C}$ -module categories are defined analogously.*

*ii) Let  $(F, f) : {}_e\mathcal{M} \rightarrow {}_e\mathcal{N}$  and  $(G, g) : {}_e\mathcal{M} \rightarrow {}_e\mathcal{N}$  be module functors. A module natural transformation  $\eta : F \rightarrow G$  is a natural transformation for which the diagrams*

$$\begin{array}{ccc} F(c \triangleright m) & \xrightarrow{\eta(c \triangleright m)} & G(c \triangleright m) \\ \downarrow f_{c,m} & & \downarrow g_{c,m} \\ c \triangleright F(m) & \xrightarrow{\text{id}_c \triangleright \eta(m)} & c \triangleright G(m), \end{array} \quad (2.21)$$

*commute for all possible objects. The category of module functors from  ${}_e\mathcal{M}$  to  ${}_e\mathcal{N}$  and module natural transformations between them is denoted by  $\text{Fun}_e({}_e\mathcal{M}, {}_e\mathcal{N})$ .*

It is easy to see that the adjoint functor of a module functor is again a module functor. The module functor constraint is uniquely determined by the requirement that the evaluation and coevaluation of the adjunction are module natural transformations. Two module categories  ${}_e\mathcal{M}$  and  ${}_e\mathcal{N}$  over  $\mathcal{C}$  are called equivalent if there exist module functors  $(F, f) : {}_e\mathcal{M} \rightarrow {}_e\mathcal{N}$  and  $(G, g) : {}_e\mathcal{N} \rightarrow {}_e\mathcal{M}$  and module natural isomorphisms  $F \circ G \rightarrow \text{id}_{{}_e\mathcal{N}}$  and  $G \circ F \rightarrow \text{id}_{{}_e\mathcal{M}}$ . The 2-category of left module categories over  $\mathcal{C}$ , module functors and module natural transformations between them is called  $\mathbf{Mod}(\mathcal{C})$ .

There is an obvious notion of a submodule category and of a direct sum of module categories. A module category is called indecomposable if it is not equivalent to a direct sum of two non trivial module categories, and it is called irreducible if it has no nontrivial submodule categories. It is shown in [17, Lemma 1] that a module category  $\mathcal{M}$  over  $\mathcal{C}$  is indecomposable if and only if it is irreducible and that in this case there are finitely many isomorphism classes of simple objects in  $\mathcal{M}$ . In particular, there exists a complement for every submodule category.

The category of modules over a separable algebra  $A \in \mathcal{C}$  is a  $\mathcal{C}$ -module category by equation (2.7). It is indecomposable if and only if the algebra is indecomposable [17, Remark 5]. The following theorem leads to the notion of Morita equivalence of fusion categories.

**Theorem 2.13** ([7, 16]) *Let  ${}_e\mathcal{M}$  be a indecomposable left  $\mathcal{C}$ -module category. The category of  $\mathcal{C}$ -module functors  $\text{Fun}_e({}_e\mathcal{M}, {}_e\mathcal{M})$  is a fusion category with monoidal structure given by composition of functors and duality by the adjunction of module functors.*

$\text{Fun}_e({}_e\mathcal{M}, {}_e\mathcal{M})$  is called the category dual to  $\mathcal{C}$  with respect to  ${}_e\mathcal{M}$ . In particular, all module natural isomorphisms from the identity functor of an indecomposable module category to itself are multiples of the identity.

**Definition 2.14** ([13]) *Suppose  $\mathcal{M}_e$  is a right  $\mathcal{C}$ -module category,  ${}_e\mathcal{N}$  a left  $\mathcal{C}$ -module category and  $\mathcal{A}$  an additive category.*

*i) A functor  $F : \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{A}$  is called  $\mathcal{C}$ -balanced if it is equipped with natural isomorphisms  $f_{m,c,n} : F(m \triangleleft c, n) \rightarrow F(m, c \triangleright n)$  for all objects  $c \in \mathcal{C}$ ,  $m \in \mathcal{M}$  and  $n \in \mathcal{N}$ ,*

such that the pentagons

$$\begin{array}{ccc}
& F(m \triangleleft (c \otimes d), n) & \\
\swarrow & & \searrow f_{m, c \otimes d, n} \\
F((m \triangleleft c) \triangleleft d, n) & & F(m, (c \otimes d) \triangleright n) \\
\downarrow f_{m \triangleleft c, d, n} & & \downarrow \\
F(m \triangleleft c, d \triangleright n) & \xrightarrow{f_{m, c, d \triangleright n}} & F(m, c \triangleright (d \triangleright n)),
\end{array} \tag{2.22}$$

commute for all possible objects. The unlabelled lines are the isomorphisms obtained from the module constraints of  $\mathbb{M}$  and  $\mathbb{N}$ , respectively. The natural isomorphism  $f$  is called balancing constraint.

ii) Let  $F, G : \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{A}$  be two  $\mathcal{C}$ -balanced functors with balancing constraints  $f$  and  $g$ , respectively. A  $\mathcal{C}$ -balanced natural transformation  $\eta : F \rightarrow G$  is a natural transformation, such that the diagrams

$$\begin{array}{ccc} F(m \triangleleft c, n) & \xrightarrow{\eta(m \triangleleft c, n)} & G(m \triangleleft c, n) \\ \downarrow f_{m, c, n} & & \downarrow g_{m, c, n} \\ F(m, c \triangleright n) & \xrightarrow{\eta(m, c \triangleright n)} & G(m, c \triangleright n) \end{array} \quad (2.23)$$

commute for all possible objects.

### 3 Module Traces

In this section we introduce a graphical calculus for module categories and derive a graphical description of the algebra morphism of the inner hom objects. In the second subsection we introduce module traces and discuss their basic properties and some examples.

### 3.1 Graphical Calculus for Module Categories

We extend the graphical calculus for tensor categories (see Appendix A) to module categories. We represent objects, morphisms and the action on a module category  ${}_c\mathcal{M}$  as follows.

$$m \hat{=} m, \quad g : m \rightarrow n \hat{=} \boxed{\phantom{g}} g, \quad c \triangleright m \hat{=} c \quad \left| \quad m \right. . \quad (3.1)$$

Any module category is equivalent to a strict module category, see [12, Thm. 1.3.8.]. This implies that the graphical notation for module categories has properties analogous to the graphical notation for tensor categories: Once parentheses and actions of unit objects are specified for the incoming and outgoing objects, each diagram unambiguously represents a morphism in  $\mathcal{M}$ .

We briefly summarise the definition of the inner hom object from [17]. Let  $\mathcal{M}$  be a left  $\mathcal{C}$ -module category. An inner hom object  ${}_{\mathcal{C}}\langle m, n \rangle^{\mathcal{M}} \in \mathcal{C}$  for  $m, n \in \mathcal{M}$  is an object in  $\mathcal{C}$  with a natural isomorphism

$$\alpha : \mathrm{Hom}_{\mathcal{M}}(c \triangleright m, n) \simeq \mathrm{Hom}_{\mathcal{C}}(c, {}_c\langle m, n \rangle^{\mathcal{M}}), \quad (3.2)$$



for all  $c \in \mathcal{C}$  and  $m, n \in \mathcal{M}$ . We write  $\langle \_, \_ \rangle$  when the relevant module category  $\mathcal{M}$  is clear from the context. Inner hom objects always exist, are unique up to a unique isomorphism and determine a bifunctor  $\langle \_, \_ \rangle^{\mathcal{M}} : \mathcal{M} \times \mathcal{M}^{\text{op}} \rightarrow \mathcal{C}$ . In the following we will speak of “the inner hom object”. The inner hom bifunctor is compatible with the module structure [17]:

$${}_c\langle m, c \triangleright n \rangle \simeq c \otimes {}_c\langle m, n \rangle, \quad \text{and} \quad {}_c\langle c \triangleright m, n \rangle \simeq {}_c\langle m, n \rangle \otimes c^*. \quad (3.3)$$

The inner hom object is represented by the following diagram:

[illegible]

and the isomorphism (3.2) reads:

$$\alpha : \begin{array}{c} c \\ | \\ \boxed{\phantom{x}} \\ | \\ n \end{array} \xrightarrow{\sim} \begin{array}{c} c \\ | \\ \boxed{\phantom{x}} \\ | \\ n \quad m \end{array} . \quad (3.5)$$

This can be visualised by flipping the string representing  $m$  and zipping it with the  $n$ -string. For a morphism  $g : n \rightarrow \tilde{n}$ , the morphism  ${}_e\langle m, g \rangle : {}_e\langle m, n \rangle \rightarrow {}_e\langle m, \tilde{n} \rangle$  is given by the diagram

$$\begin{array}{c}
n \\
\hline
m \\
\hline
g \\
\hline
\tilde{n}
\end{array}
. \quad (3.6)$$

Each morphism  $h : m \rightarrow \tilde{m}$  defines a morphism  ${}_e\langle h, n \rangle : {}_e\langle \tilde{m}, n \rangle \rightarrow {}_e\langle m, n \rangle$  that is depicted as

$$\begin{array}{c} \text{---} \\ | \\ n \\ | \\ \tilde{m} \\ | \\ \square \quad h^* \\ | \\ m \\ \text{---} \end{array} \quad . \quad (3.7)$$

The symbol  $h^*$  indicates that the functor  ${}_c\langle \cdot, \cdot \rangle$  is contravariant in the first argument.

**Remark 3.1** In the case of  $\mathcal{C}$  considered as a left module category over itself, the inner hom object of  $c, d \in \mathcal{C}$  is given by  ${}_c\langle c, d \rangle = d \otimes c^*$ . For a morphism  $h : c \rightarrow \tilde{c}$  indeed  ${}_c\langle h, d \rangle = \text{id}_d \otimes h^*$ . The notation  $h^*$  therefore is consistent.

The naturality of  $\alpha : \text{Hom}_{\mathcal{M}}(c \triangleright m, n) \simeq \text{Hom}_{\mathcal{C}}(c, {}_c\langle m, n \rangle)$  manifests itself in the graphical calculus as follows:

*i)*  $\alpha$  is natural with respect to  $m$ :

$$\alpha : \begin{array}{c} \text{---} \\ | \\ c \\ \square \\ | \\ h \\ | \\ m \\ \square \\ | \\ n \end{array} \mapsto \begin{array}{c} \text{---} \\ | \\ c \\ \square \\ | \\ m \\ | \\ n \\ \square \\ | \\ h^* \\ | \\ \tilde{m} \end{array} = \begin{array}{c} \text{---} \\ | \\ c \\ \square \\ | \\ m \\ \square \\ | \\ \alpha(h) \\ | \\ \tilde{m} \\ | \\ n \end{array}. \quad (3.8)$$

ii)  $\alpha$  is natural with respect to  $n$ :

$$\alpha : \begin{array}{c} c \\ | \\ \boxed{\phantom{f}} \\ | \\ n \\ | \\ g \\ | \\ \tilde{n} \end{array} f \mapsto \begin{array}{c} c \\ | \\ \boxed{\phantom{f}} \\ | \\ n \\ | \\ g \\ | \\ \tilde{n} \end{array} \alpha(f) = \begin{array}{c} c \\ | \\ \boxed{\phantom{f}} \\ | \\ \tilde{n} \\ | \\ m \end{array} \alpha(g \circ f) . \quad (3.9)$$

iii)  $\alpha$  is natural with respect to  $c$ :

$$\alpha : \begin{array}{c} c \\ | \\ \gamma \\ | \\ d \\ | \\ \boxed{\phantom{f}} \\ | \\ n \end{array} f \mapsto \begin{array}{c} c \\ | \\ \gamma \\ | \\ d \\ | \\ \boxed{\phantom{f}} \\ | \\ n \end{array} \alpha(f) = \begin{array}{c} c \\ | \\ \boxed{\phantom{f}} \\ | \\ n \\ | \\ m \end{array} \alpha(f \circ (\gamma \triangleright m)) . \quad (3.10)$$

**Lemma 3.2** *The natural isomorphism  $\alpha$  from equation (3.2) is compatible with the module structure. For all morphisms  $\gamma : x \rightarrow y$  in  $\mathcal{C}$  and all  $f \in \text{Hom}(c \triangleright m, n)$ ,*

$$\begin{array}{c} x \\ | \\ \gamma \\ | \\ y \end{array} \begin{array}{c} c \\ | \\ \boxed{\phantom{f}} \\ | \\ n \end{array} f \xrightarrow{\alpha} \begin{array}{c} x \\ | \\ \gamma \\ | \\ y \end{array} \begin{array}{c} c \\ | \\ \boxed{\phantom{f}} \\ | \\ n \end{array} \alpha(f) . \quad (3.11)$$

*Proof:* It suffices to prove the statement for  $y = x$  and  $\gamma = \text{id}_x$ . The general case then follows directly from the naturality of  $\alpha$ . First recall that the canonical isomorphism  ${}_e\langle m, c \triangleright n \rangle \simeq c \otimes {}_e\langle m, n \rangle$  is constructed as follows. Consider for  $x, c \in \mathcal{C}$  and  $m, n \in \mathcal{M}$  the square:

$$\begin{array}{ccc} \text{Hom}(x \triangleright (c \triangleright m), n) & \xrightarrow{\simeq} & \text{Hom}(c \triangleright m, {}^*x \triangleright n) \\ \downarrow \simeq & & \downarrow \alpha \\ \text{Hom}((x \otimes c) \triangleright m, n) & & \text{Hom}(c, {}_e\langle m, {}^*x \triangleright n \rangle) \\ \downarrow \alpha & & \downarrow \simeq \\ \text{Hom}(x \otimes c, {}_e\langle m, n \rangle) & \xrightarrow{\simeq} & \text{Hom}(c, {}^*x \otimes {}_e\langle m, n \rangle) . \end{array} \quad (3.12)$$

The horizontal isomorphisms are induced by the duality in  $\mathcal{C}$ , while the unlabelled vertical isomorphism on the right is the natural isomorphism  ${}_e\langle m, {}^*x \triangleright n \rangle \simeq {}^*x \otimes {}_e\langle m, n \rangle$  from equation (3.3). This isomorphism is defined by the requirement that the square commutes. As it is constructed from natural isomorphisms which we suppress in the graphical notation, we will suppress this isomorphism as well in the sequel. It follows from the commutativity of the previous diagram, that the diagram

$$\begin{array}{ccc} \text{Hom}(x \triangleright (c \triangleright m), x \triangleright n) & \xrightarrow{\simeq} & \text{Hom}(x^* \otimes x \otimes c \triangleright m, n) \\ \downarrow \alpha & & \downarrow \alpha \\ \text{Hom}(x \otimes c, x \otimes {}_e\langle m, n \rangle) & \xleftarrow{\simeq} & \text{Hom}((x^* \otimes x \otimes c, {}_e\langle m, n \rangle) \end{array} \quad (3.13)$$

commutes. If we choose  $\text{id}_x \otimes f \in \text{Hom}(x \triangleright (c \triangleright m), x \triangleright n)$  with  $f \in \text{Hom}(c \triangleright m, n)$  in the left upper space, the commutativity of the diagram implies  $\alpha(\text{id}_x \otimes f) = \text{id}_x \otimes \alpha(f)$ .  $\square$

We will now develop a graphical representation for the internal multiplication and show that it equips  ${}_e\langle m, m \rangle$  with the structure of an algebra. The internal evaluation morphism  $\text{ev}_{n,m} : {}_e\langle n, m \rangle \triangleright n \rightarrow m$  (see [17, Sec. 3.2.]) is given by:

$$\text{ev}_{n,m} = \alpha^{-1}(\text{id}_{{}_e\langle n, m \rangle}) \hat{=} \begin{array}{c} \text{---} n \text{---} \\ \text{---} m \text{---} \end{array} . \quad (3.14)$$

This notation is compatible with the notation for  $\alpha$  since by flipping the  $n$ -string we obtain the identity string  ${}_e\langle n, m \rangle$ . The internal multiplication  $\mu_{m,n,k} : {}_e\langle n, k \rangle \otimes {}_e\langle m, n \rangle \rightarrow {}_e\langle m, k \rangle$  and the internal unit  $\eta_m : 1 \rightarrow {}_e\langle m, m \rangle$  are given by

$$\mu_{m,n,k} = \alpha \left( \begin{array}{c} \text{---} m \text{---} \\ \text{---} n \text{---} \\ \text{---} k \text{---} \end{array} \right) \hat{=} \begin{array}{c} \text{---} n \text{---} \\ \text{---} k \text{---} \\ \text{---} m \text{---} \end{array} , \quad \eta_m = \alpha \left( \begin{array}{c} \text{---} m \text{---} \\ \text{---} m \text{---} \end{array} \right) \hat{=} \begin{array}{c} \bullet \\ \text{---} m \text{---} \end{array} . \quad (3.15)$$

**Lemma 3.3** *For all morphisms  $f \in \text{Hom}(c \triangleright m, {}_e\langle n, k \rangle \triangleright n)$ ,*

$$\alpha(\text{ev}_{n,k} \circ f) = \mu_{m,n,k} \circ \alpha(f), \text{ i.e.} \quad (3.16)$$

$$\begin{array}{c} c \text{---} m \\ \text{---} f \text{---} \\ \text{---} n \text{---} \\ \text{---} k \text{---} \end{array} \xrightarrow{\alpha} \begin{array}{c} c \text{---} \\ \text{---} \alpha(f) \text{---} \\ \text{---} n \text{---} \\ \text{---} k \text{---} \\ \text{---} m \text{---} \end{array} . \quad (3.17)$$

*Proof:* The identity

$$\begin{array}{c} d \text{---} l \\ \text{---} g \text{---} \\ \text{---} s \text{---} \end{array} = \alpha(g) \begin{array}{c} d \text{---} \\ \text{---} \alpha(g) \text{---} \\ \text{---} l \text{---} \\ \text{---} s \text{---} \end{array} , \quad (3.18)$$

for all  $g \in \text{Hom}(d \triangleright l, s)$  follows from applying  $\alpha$  to both sides and using the naturality of  $\alpha$ . Applying this identity to  $f$  with  $s = {}_e\langle n, k \rangle \triangleright n$  yields

$$\begin{array}{c} c \text{---} m \\ \text{---} f \text{---} \\ \text{---} n \text{---} \\ \text{---} k \text{---} \end{array} = \alpha(f) \begin{array}{c} c \text{---} \\ \text{---} \alpha(f) \text{---} \\ \text{---} m \text{---} \\ \text{---} n \text{---} \\ \text{---} k \text{---} \end{array} . \quad (3.19)$$

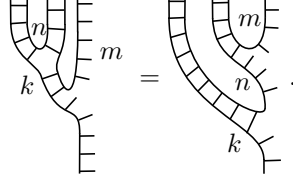
Applying  $\alpha$  to the right hand side of this equation and using its naturality proves the claim.  $\square$

The following theorem plays an important role in the theory of fusion categories since it combines the theory of module categories with the theory of algebras.

**Theorem 3.4** ([17]) *For all non-zero objects  $m, n$  in a  $\mathcal{C}$ -module category  ${}_e\mathcal{M}$ ,  ${}_e\langle m, m \rangle$  is an algebra object in  $\mathcal{C}$  and  ${}_e\langle m, n \rangle$  is a right  ${}_e\langle m, m \rangle$ -module. The functor  $\mathcal{M} \ni n \rightarrow {}_e\langle m, n \rangle \in \text{Mod}_{\mathcal{C}}({}_e\langle m, m \rangle)$  yields an equivalence of  $\mathcal{C}$ -module categories provided  ${}_e\mathcal{M}$  is indecomposable.*

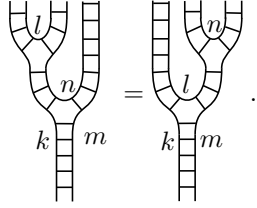
We will revisit parts of the proof of this statement with the graphical calculus.

**Proposition 3.5** *i) The internal evaluation morphism is a module morphism:*



$$(3.20)$$

*ii) The internal multiplication is associative:*



$$(3.21)$$

*iii) For all non-zero  $m \in \mathcal{M}$ ,  ${}_e\langle m, m \rangle$  is canonically an algebra object.*

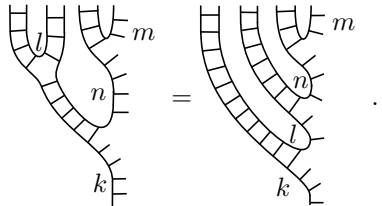
*Proof:* The first relation follows from applying  $\alpha$  to both diagrams. Both diagrams obtained in this way represent the multiplication morphism. Since  $\alpha$  is an isomorphism, the preimages have to agree as well.

To show the second part, first note that the expression on the left hand side of equation (3.21) is  $\alpha$  applied to



$$(3.22)$$

Part *i)* implies



$$(3.23)$$

Now apply  $\alpha$  to the diagram on the right. In the upper part of the diagram this results in the morphism  $\text{id}_{{}_e\langle l, k \rangle} \otimes \mu_{m, n, l}$  due to Lemma 3.2. With Lemma 3.3 we conclude that  $\alpha$  applied to this diagram yields the right hand side of equation (3.21). The statement follows since  $\alpha$  is an isomorphism. To show the last part we only have to prove the compatibility of the internal multiplication and the internal unit. This is a direct computation in the diagrammatic calculus.  $\square$

## 3.2 Module Traces on Module Categories over Pivotal Fusion Categories

We are now ready to define the notion of a module trace. As example we discuss module categories over  $G$ -graded vector spaces. This illustrates that the existence of a module trace on a given module category distinguishes different pivotal structures.

For each module category  $\mathcal{M}$  over a pivotal category  $\mathcal{C}$  there is a linear map

$$\mathrm{tr}_{c,m}^{\mathcal{C}} : \mathrm{End}_{\mathcal{M}}(c \triangleright m) \rightarrow \mathrm{End}_{\mathcal{M}}(m), \quad f \mapsto (\mathrm{ev}'_c \triangleright \mathrm{id}_m) \circ (a*_c \triangleright f) \circ (\mathrm{coev}_c \triangleright \mathrm{id}_m), \quad (3.24)$$

which we call partial trace. Whenever this is unambiguous we omit the labels of  $\mathrm{tr}^{\mathcal{C}}$ . The graphical representation of this map is

$$\mathrm{tr}^{\mathcal{C}} \left( \begin{array}{c} \text{diagram of } f \text{ on } c \triangleright m \end{array} \right) = \begin{array}{c} \text{diagram of } \mathrm{tr}^{\mathcal{C}}(f) \text{ on } m \end{array}. \quad (3.25)$$

As a direct consequence of the definition of a module functor we obtain:

**Lemma 3.6** *Let  $F : {}_{\mathcal{C}}\mathcal{M} \rightarrow {}_{\mathcal{C}}\mathcal{N}$  be a  $\mathcal{C}$ -module functor. For all  $f \in \mathrm{End}_{\mathcal{M}}(c \triangleright m)$ ,  $\mathrm{tr}^{\mathcal{C}}(F(f)) = F(\mathrm{tr}^{\mathcal{C}}(f))$ .*

With the map  $\mathrm{tr}^{\mathcal{C}}$  we can define module traces.

**Definition 3.7** *Let  $\mathcal{M}$  be a module category over a pivotal fusion category  $\mathcal{C}$ . A trace  $\Theta$  on  $\mathcal{M}$  is a collection of linear maps*

$$\Theta_m : \mathrm{End}_{\mathcal{M}}(m) \rightarrow \mathbb{C} \quad \text{for all } m \in \mathcal{M}, \quad (3.26)$$

*such that the following properties are satisfied:*

i)  $\Theta$  is symmetric: for all  $f \in \mathrm{Hom}_{\mathcal{M}}(m, n)$  and  $g \in \mathrm{Hom}_{\mathcal{M}}(n, m)$ ,

$$\Theta_m(g \circ f) = \Theta_n(f \circ g). \quad (3.27)$$

ii)  $\Theta$  is non-degenerate: the pairing

$$\mathrm{Hom}_{\mathcal{M}}(m, n) \times \mathrm{Hom}_{\mathcal{M}}(n, m) \rightarrow \mathbb{C}, \quad (f, g) \mapsto \Theta_m(g \circ f) \quad (3.28)$$

*is non-degenerate for all  $m, n \in \mathcal{M}$ .*

*If furthermore*

iii)  $\Theta$  is  $\mathcal{C}$ -compatible: for all  $c \in \mathcal{C}$ ,  $m \in \mathcal{M}$ ,

$$\Theta_{c \triangleright m} = \Theta_m \circ \mathrm{tr}^{\mathcal{C}}, \quad (3.29)$$

*then  $\Theta$  is called a  $\mathcal{C}$ -module trace or module trace if the category  $\mathcal{C}$  is clear from the context. We sometimes write  $(\mathcal{M}_{\mathcal{C}}, \Theta^{\mathcal{M}})$  for a module category with module trace.*

The notion of a trace on a linear category is well-known and a category with a trace is also called a Calabi-Yau category, see e.g. [4, Sec. 2].

**Remark 3.8** i) The notion of a module trace is a generalisation of the trace on a pivotal fusion category. Indeed, consider  $\mathcal{C}$  as a left module category over itself. The right trace  $\mathrm{tr}^R : \mathrm{End}_{\mathcal{C}}(c) \rightarrow \mathbb{C}$  induces a canonical module trace on  $\mathcal{C}$ . The right trace is symmetric and the compatibility of the duality with the tensor product yields equation (3.29). For the non-degeneracy, note that the argument in the proof of [20, Lemma II.4.2.3] can be extended to the case of pivotal fusion categories, see also Lemma 5.1. The semisimplicity of  $\mathcal{C}$  is crucial at this point and for this reason we restrict attention to fusion categories and do not consider more general tensor categories.

- ii) For any given trace  $\Theta$  on  $\mathcal{M}$  and non-zero number  $z \in \mathbb{C}$  the linear maps  $z \cdot \Theta_m$  define another trace denoted  $z \cdot \Theta$ . If  $\Theta$  is a module trace then  $z \cdot \Theta$  is again a module trace. We will show in Section 4.1 that module traces are unique up to such rescalings.

We introduce a graphical notation for module traces:

$$\Theta_m(f) \cong \begin{array}{c} \text{---} m \\ | \\ \text{---} f \\ | \\ \text{---} n \end{array} . \quad (3.30)$$

The symmetry and  $\mathcal{C}$ -compatibility of  $\Theta$  then read

$$\begin{array}{c} \text{---} m \\ | \\ \text{---} f \\ | \\ \text{---} n \\ | \\ \text{---} g \\ | \\ \text{---} m \end{array} = \begin{array}{c} \text{---} n \\ | \\ \text{---} g \\ | \\ \text{---} m \\ | \\ \text{---} f \\ | \\ \text{---} n \end{array} , \quad \begin{array}{c} \text{---} m \\ | \\ \text{---} c \\ | \\ \text{---} m \end{array} = \begin{array}{c} \text{---} m \\ | \\ \text{---} c \\ | \\ \text{---} m \end{array} . \quad (3.31)$$

Given a trace  $\Theta$  on a category  $\mathcal{M}$ , we define the dimensions of objects  $m \in \mathcal{M}$  with respect to  $\Theta$  as

$$\dim^\Theta(m) = \Theta_m(\text{id}_m). \quad (3.32)$$

The dimensions depend only on the isomorphism classes of objects:

- Lemma 3.9** *i) If two objects  $m, n \in \mathcal{M}$  are isomorphic then  $\dim^\Theta(m) = \dim^\Theta(n)$ .  
ii)  $\Theta$  is compatible with direct sums. For all  $f \in \text{End}_{\mathcal{M}}(m)$ ,  $g \in \text{End}_{\mathcal{M}}(n)$ , we have  $\Theta_{m \oplus n}(f \oplus g) = \Theta_m(f) + \Theta_n(g)$ . In particular,  $\dim^\Theta(m \oplus n) = \dim^\Theta(m) + \dim^\Theta(n)$ .  
iii)*

$$\dim^\Theta(c \triangleright m) = \dim^c(c) \cdot \dim^\Theta(m). \quad (3.33)$$

*Proof:* For the first part choose an isomorphism  $f : m \rightarrow n$ . The symmetry of  $\Theta$  implies

$$\dim^\Theta(m) = \Theta_m(\text{id}_m) = \Theta_m(f^{-1} \circ f) = \Theta_n(f \circ f^{-1}) = \Theta_n(\text{id}_n) = \dim^\Theta(n). \quad (3.34)$$

The second part follows directly from the linearity of  $\Theta$ . The third part is a consequence of the  $\mathcal{C}$ -compatibility of  $\Theta$ .  $\square$

### Direct Sums and Equivalences of Module Categories with Module Trace

We show that the notion of a module trace is well-behaved with respect to decomposition of module categories and investigate the structure of the module categories with  $\mathcal{C}$ -module trace in the 2-category  $\text{Mod}(\mathcal{C})$  of  $\mathcal{C}$ -module categories, module functors and module natural transformations.

**Definition 3.10** *Let  $\text{Mod}^\Theta(\mathcal{C})$  be the full sub 2-category of  $\text{Mod}(\mathcal{C})$  which has  $\mathcal{C}$ -module categories  $(\mathcal{M}_c, \Theta^{\mathcal{M}})$  endowed with a  $\mathcal{C}$ -module trace  $\Theta^{\mathcal{M}}$  as objects. A module functor  $F : {}_c\mathcal{M} \rightarrow {}_c\mathcal{N}$  is called an isometric module functor if  $\Theta^{\mathcal{N}}(F(f)) = \Theta^{\mathcal{M}}(f)$  for all  $f \in \text{End}_{\mathcal{M}}(m)$  and all  $m \in \mathcal{M}$ . Two module categories in  $\text{Mod}^\Theta(\mathcal{C})$  are called isometrically equivalent if there exists an equivalence of module categories consisting of isometric module functors between them.*

Note that an isometric module functor is faithful due to the non-degeneracy of the module traces. The subcategory  $\text{Mod}^\Theta(\mathcal{C})$  is well-behaved in the following sense.

- Proposition 3.11** *i) Let  $({}_e\mathcal{M}, \Theta^{\mathcal{M}})$  be a an object in  $\text{Mod}^{\Theta}(\mathcal{C})$  and let  ${}_e\mathcal{N}$  be a module category with an equivalence  $F : {}_e\mathcal{N} \rightarrow {}_e\mathcal{M}$  of module categories. Then there exists a  $\mathcal{C}$ -module trace on  ${}_e\mathcal{N}$  such that  $F$  is an isometric equivalence.*
- ii) The direct sum of two module categories with module traces possesses a canonical module trace.*
- iii) A submodule category of a module category with module trace inherits a canonical module trace.*
- iv) Each object in  $\text{Mod}^{\Theta}(\mathcal{C})$  is isometrically equivalent to a finite direct sum of indecomposable objects.*

*Proof:* To show the first part, define the linear maps  $\Theta^{\mathcal{N}}(f) = \Theta^{\mathcal{M}}(F(f))$  for all  $f \in \text{End}_{\mathcal{N}}(n)$ . Lemma 3.6 implies that this defines a module trace for  $\mathcal{N}$  and that  $F$  is isometric by construction. For the second part consider an object  $m \oplus n \in {}_e\mathcal{M} \oplus {}_e\mathcal{N}$ . Since  $\text{End}_{\mathcal{M} \oplus \mathcal{N}}(m \oplus n) = \text{End}_{\mathcal{M}}(m) \oplus \text{End}_{\mathcal{N}}(n)$ , we can define a linear map  $(\Theta^{\mathcal{M}} \oplus \Theta^{\mathcal{N}})_{m \oplus n} : \text{End}_{\mathcal{M} \oplus \mathcal{N}}(m \oplus n) \rightarrow \mathbb{C}$  as the sum  $\Theta_m^{\mathcal{M}} \oplus \Theta_n^{\mathcal{N}}$ . It is easy to see that this defines a  $\mathcal{C}$ -module trace. Now consider a submodule category of a module category with module trace. As we can choose a complement of the submodule category, the restriction of a module trace to a submodule category is non-degenerate and hence a module trace. The last statement is a consequence of the first and second statement.  $\square$

### Examples

We denote by  $\mathbf{Vect}$  the fusion category of finite dimensional  $\mathbb{C}$ -vector spaces. A semisimple abelian category over  $\mathbb{C}$  is a module category over  $\mathbf{Vect}$  with module structure  $V \otimes_{\mathbb{C}} m$  defined by  $V \otimes_{\mathbb{C}} \text{Hom}_{\mathcal{M}}(m, n) \simeq \text{Hom}_{\mathcal{M}}(V \otimes_{\mathbb{C}} m, n)$  for  $V \in \mathbf{Vect}$  and  $m, n \in \mathcal{M}$ .

**Lemma 3.12** *A trace on a semisimple category  $\mathcal{M}$  is also a  $\mathbf{Vect}$ -module trace on  $\mathcal{M}$ .*

*Proof:* We show that condition (3.29) is satisfied. As  $\text{End}_{\mathcal{M}}(V \otimes_{\mathbb{C}} m) \simeq \text{End}(V) \otimes \text{End}(m)$ , it is sufficient to show that

$$\Theta_{V \otimes_{\mathbb{C}} m}(\alpha \otimes_{\mathbb{C}} f) = \text{tr}(\alpha) \Theta_m(f) \quad (3.35)$$

for all  $V \in \mathbf{Vect}$ ,  $\alpha \in \text{End}(V)$  and  $f \in \text{End}(m)$ . Here  $\text{tr}$  is the usual trace on  $\mathbf{Vect}$  that coincides with the right trace on  $\mathbf{Vect}$  considered as a fusion category. Equation (3.35) follows from a direct calculation in a basis for  $V$ .  $\square$

**Example 3.13** Let  $G$  be a finite group and  $\omega \in \mathcal{C}^3(G, \mathbb{C}^{\times})$  a normalised cocycle. This data defines a fusion category  $\mathbf{Vect}_G^{\omega}$  with simple objects labelled by elements of  $G$ , see [6] and [17] for more details. The pivotal structures on  $G$  are in bijection with the characters  $\kappa \in \text{Hom}(G, \mathbb{C}^{\times})$ . Indecomposable module categories  $M(H, \Psi)$  over  $\mathbf{Vect}_G^{\omega}$  are obtained from subgroups  $H \subset G$  with  $\omega|_H = 1$  and cocycles  $\Psi \in \mathcal{C}^2(H, \mathbb{C}^{\times})$ . The simple objects of  $M(H, \Psi)$  are labelled by elements in the right cosets  $[g] \in H \backslash G$ . The action of a simple object  $x \in \mathbf{Vect}_G^{\omega}$  is given by  $x \triangleright [g] = [xg]$ , with module constraint twisted by  $\Psi$ .

A module category  $M(H, \Psi)$  over the pivotal fusion category  $(\mathbf{Vect}_G^{\omega}, \kappa)$  possesses a module trace if and only if  $\kappa|_H = 1$ . This can be seen as follows: Suppose  $\Theta$  is a module trace on  $M(H, \Psi)$  normalised by  $\Theta([e]) = 1$ . Then equation (3.33) implies  $\Theta([gx]) = \kappa(g) \cdot \Theta([x])$ , in particular  $\Theta([g]) = \kappa(g)$ . So  $\kappa$  is well-defined on  $H \backslash G$ , which is the case if and only if  $\kappa|_H = 1$ . Conversely, if  $\kappa|_H = 1$  it is easy to see that  $\kappa$  yields a module trace for  $M(H, \Psi)$ . In particular, there exists a module trace for all module categories over  $\mathbf{Vect}_G$  when the pivotal structure  $\kappa \equiv 1$  is chosen.

**Example 3.14** Let  $\mathcal{C}$  be a fusion category. Recall the construction of a pivotal fusion category  $\tilde{\mathcal{C}}$  from [7, Remark 3.1]: The simple objects of  $\tilde{\mathcal{C}}$  are pairs  $(c, f_c)$ , where  $c \in \mathcal{C}$

is a simple object and  $f_c : c \rightarrow c^{**}$  is an isomorphism such that  $f_c^{**}f_c = g_c$ , where  $g$  is the canonical monoidal natural isomorphism  $\text{id}_{\mathcal{C}} \rightarrow (\cdot)^{****}$  defined in [7]. With  $(c, f_c) \in \tilde{\mathcal{C}}$ , also  $(c, -f_c) \in \tilde{\mathcal{C}}$ .  $\tilde{\mathcal{C}}$  has a canonical pivotal structure such that  $\dim^{\tilde{\mathcal{C}}}(c, f_c) = \text{ev}'_{c^{**}} \circ (f_c \otimes \text{id}_{c^*}) \circ \text{coev}_c =: \text{tr}(f_c)$ . The monoidal structure of  $\tilde{\mathcal{C}}$  is induced by the monoidal structure of  $\mathcal{C}$  and the forgetful functor  $U : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$  is a monoidal functor. Hence  $\mathcal{C}$  is a left  $\tilde{\mathcal{C}}$ -module category. This module category does not admit a module trace when  $\tilde{\mathcal{C}}$  is equipped with the canonical pivotal structure: Assume that  $\Theta$  is a module trace and let  $(c, f_c) \in \tilde{\mathcal{C}}$  and  $d \in \mathcal{C}$  be simple objects. Then

$$\begin{aligned} \text{tr}(f_c) \dim^{\Theta}(d) &= \dim^{\Theta}((c, f_c) \triangleright d) = \dim^{\Theta}(c \otimes d) \\ &= \dim^{\Theta}((c, -f_c) \triangleright d) = -\text{tr}(f_c) \dim^{\Theta}(d), \end{aligned} \tag{3.36}$$

which is a contradiction, since as we will explain in Section 5,  $\dim^{\Theta}(d) \neq 0$ .

However, a pivotal structure  $a$  for  $\mathcal{C}$  induces a different pivotal structure for  $\tilde{\mathcal{C}}$  with quantum dimensions  $\dim^{(\tilde{\mathcal{C}}, a)}(c, f_c) = \dim^a(c)$  and it is easy to see that the right trace with respect to  $a$  defines a  $\tilde{\mathcal{C}}$ -module trace for the module category  $\mathcal{C}$ .

These examples motivate the following definition.

**Definition 3.15** *Let  $\mathcal{C}$  be a fusion category with pivotal structure  $a$  and  ${}_e\mathcal{M}$  a module category. The pair  $(a, {}_e\mathcal{M})$  is called matched if there exists a  $\mathcal{C}$ -module trace on  ${}_e\mathcal{M}$ . A pivotal structure for  $\mathcal{C}$  that is matched with all module categories is called flexible.*

In Proposition 5.8 we will show that a pseudo-unitary  $\mathcal{C}$  admits a flexible pivotal structure that is also spherical. It has been conjectured in [7] that all fusion categories admit a pivotal structure. The theory of module traces raises the following refinements of this question.

- Given a fusion category  $\mathcal{C}$  and an indecomposable module category  ${}_e\mathcal{M}$ , is there a pivotal structure  $a$  on  $\mathcal{C}$ , such that the pair  $(a, {}_e\mathcal{M})$  is matched?
- Does every (modular) fusion categories exhibit a flexible pivotal structure and is it unique?
- Is every flexible pivotal structure spherical?

## 4 Module Traces and the Dual Hom-Spaces

### 4.1 Uniqueness of Module Traces

In this subsection we show that module traces are unique up to scaling. First we examine traces on abelian categories and give an equivalent characterisation of traces in terms of certain natural isomorphisms. In the next step we show that analogous results hold for module traces.

We denote by  $V'$  the dual vector space of a  $\mathbb{C}$ -vector space  $V$ .

**Proposition 4.1** *Let  $\mathcal{M}$  be an additive category enriched over  $\mathbf{Vect}$ . The following structures on  $\mathcal{M}$  are in one-to-one correspondence.*

- i) A trace on  $\mathcal{M}$ .
- ii) A natural isomorphism  $\eta : \text{Hom}_{\mathcal{M}}(m, n) \rightarrow \text{Hom}_{\mathcal{M}}(n, m)'$ .



*Proof:* Let  $\Theta$  be a trace on  $\mathcal{M}$ . The non-degenerate pairing  $\text{Hom}_{\mathcal{M}}(m, n) \times \text{Hom}_{\mathcal{M}}(n, m) \rightarrow \mathbb{C}$  defines isomorphisms  $\eta_{m,n} : \text{Hom}_{\mathcal{M}}(m, n) \simeq \text{Hom}_{\mathcal{M}}(n, m)'$ . We have to show that these isomorphisms are natural, i.e. that for  $\chi : n \rightarrow \tilde{n}$  the diagram

$$\begin{array}{ccc} \text{Hom}(m, n) & \xrightarrow{\eta_{m,n}} & \text{Hom}(n, m)' \\ \downarrow \text{Hom}(m, \chi) & & \downarrow \text{Hom}(\chi, m)' \\ \text{Hom}(m, \tilde{n}) & \xrightarrow{\eta_{m,\tilde{n}}} & \text{Hom}(\tilde{n}, m)', \end{array} \quad (4.1)$$

commutes. Let  $f \in \text{Hom}(m, n)$  and  $g \in \text{Hom}(\tilde{n}, m)$ .  $\text{Hom}(m, \chi)$  is the linear map that sends  $f$  to  $\chi \circ f$ . In the following we denote this map by  $\chi$ . We compute

$$\begin{aligned} (\eta_{m,\tilde{n}} \circ \text{Hom}(m, \chi))(f)(g) &= \Theta_m(g \circ (\chi \circ f)) \\ &= \Theta_m((g \circ \chi) \circ f) = ((\text{Hom}(\chi, m)' \circ \eta_{m,n})(f))(g). \end{aligned} \quad (4.2)$$

This shows the commutativity of the diagram (4.1). The proof for naturality in  $m$  is analogous.

On the other hand, a natural isomorphism  $\eta_{m,n} : \text{Hom}_{\mathcal{M}}(m, n) \rightarrow \text{Hom}_{\mathcal{M}}(n, m)'$  induces a linear map  $\Theta_m : \text{Hom}_{\mathcal{M}}(m, m) \rightarrow \mathbb{C}$  by  $\Theta_m(f) = \eta_{m,m}(\text{id}_m)(f)$ . For  $\alpha \in \text{Hom}(m, n)$  and  $\beta \in \text{Hom}(n, m)$ , the naturality of  $\eta$  implies

$$\Theta_m(\beta \circ \alpha) = \eta_{m,m}(\text{id}_m)(\beta \circ \alpha) = \eta_{m,n}(\alpha)(\beta) = \eta_{n,n}(\alpha \circ \beta) = \Theta_n(\alpha \circ \beta). \quad (4.3)$$

This proves the symmetry of  $\Theta$  and, as the map  $\eta_{m,n}$  is an isomorphism, also the non-degeneracy.  $\square$

We will now generalise this proposition to  $\mathcal{C}$ -module traces. Let  ${}_e\mathcal{M}$  be a  $\mathcal{C}$ -left module category. The functors

$$\begin{aligned} \mathcal{M}^{\text{op}} \times \mathcal{M} &\rightarrow \text{Vect}, & m \times n &\mapsto \text{Hom}_{\mathcal{M}}(m, n) \quad \text{and} \\ \mathcal{M}^{\text{op}} \times \mathcal{M} &\rightarrow \text{Vect}, & m \times n &\mapsto \text{Hom}_{\mathcal{M}}(n, m)' \end{aligned} \quad (4.4)$$

are canonically  $\mathcal{C}$ -balanced (see Definition 2.14). The balancing constraint for the first functor is the natural isomorphism

$$\text{Hom}_{\mathcal{M}}(m, c \triangleright n) \simeq \text{Hom}_{\mathcal{M}}(c^* \triangleright m, n) = \text{Hom}_{\mathcal{M}}(m \triangleleft^{\text{op}} c, n), \quad (4.5)$$

that is available in any tensor category. In contrast, the balancing constraint for the second functor,

$$\begin{aligned} \text{Hom}_{\mathcal{M}}(c \triangleright n, m)' &= \text{Hom}_{\mathcal{M}}(n, {}^*c \triangleright m)' \\ &\simeq \text{Hom}_{\mathcal{M}}(n, m \triangleleft^{\text{op}} (**c))' \simeq \text{Hom}_{\mathcal{M}}(n, m \triangleleft^{\text{op}} c)', \end{aligned} \quad (4.6)$$

involves the pivotal structure of  $\mathcal{C}$  in the last isomorphism.

**Theorem 4.2** *Let  ${}_e\mathcal{M}$  be a left module category over a pivotal fusion category  $\mathcal{C}$ . The following structures on  $\mathcal{M}$  are in canonical one-to-one correspondence.*

- i) A  $\mathcal{C}$ -module trace on  $\mathcal{M}$ .
- ii) A  $\mathcal{C}$ -balanced natural isomorphism  $\eta : \text{Hom}_{\mathcal{M}}(m, n) \rightarrow \text{Hom}_{\mathcal{M}}(n, m)'$ .

*Proof:* We have to show that the isomorphisms  $\eta_{m,n} : \text{Hom}_{\mathcal{M}}(m, n) \rightarrow \text{Hom}_{\mathcal{M}}(n, m)'$  from Proposition 4.1 are  $\mathcal{C}$ -balanced if and only if  $\Theta$  is  $\mathcal{C}$ -compatible. Consider morphisms  $f \in \text{Hom}(m, c \triangleright n)$  and  $g \in \text{Hom}(c \triangleright n, m)$ . Denote by  $\hat{f} \in \text{Hom}(c^* \triangleright m, n)$  and  $\hat{g} \in \text{Hom}(n, c^* \triangleright m)$  the morphisms obtained from  $f$  and  $g$  under the balancing isomorphisms

(4.5) and (4.6), respectively. A direct computation shows that the  $\mathcal{C}$ -balancing property of  $\eta_{m,n}$  is equivalent to the condition

$$\Theta_m(g \circ f) = \Theta_{c^* \triangleright m}(\hat{g} \circ \hat{f}), \quad (4.7)$$

for all possible  $f$  and  $g$ . Due to the symmetry of  $\Theta$ ,  $\Theta_{c^* \triangleright m}(\hat{g} \circ \hat{f}) = \Theta_n(\text{tr}^{\mathcal{C}}(f \circ g))$ , and we conclude that equation (4.7) is equivalent to the  $\mathcal{C}$ -compatibility of  $\Theta$ . Thus the statement is proven.  $\square$

This implies in particular that for each pivotal fusion category  $\mathcal{C}$  there is a natural  $\mathcal{C}$ -balanced isomorphism

$$\eta^{\mathcal{C}} : \text{Hom}(x, y) \rightarrow \text{Hom}(y, x)', \quad (4.8)$$

induced by the right trace.

In the sequel we will need the following extension of the usual Yoneda lemma.

**Lemma 4.3** *Let  $F, G : {}_{\mathcal{C}}\mathcal{M} \rightarrow {}_{\mathcal{C}}\mathcal{N}$  be module functors. The set of  $\mathcal{C}$ -module natural transformations  $F \rightarrow G$  is in canonical bijection with the set of  $\mathcal{C}$ -balanced natural transformations of the two  $\mathcal{C}$ -balanced functors:*

$$\begin{aligned} \mathcal{N}_{\mathcal{C}}^{\text{op}} \times {}_{\mathcal{C}}\mathcal{M} \ni n \times m &\mapsto \text{Hom}_{\mathcal{N}}(n, F(m)) \in \text{Vect}, \text{ and} \\ \mathcal{N}_{\mathcal{C}}^{\text{op}} \times {}_{\mathcal{C}}\mathcal{M} \ni n \times m &\mapsto \text{Hom}_{\mathcal{N}}(n, G(m)) \in \text{Vect}. \end{aligned} \quad (4.9)$$

A  $\mathcal{C}$ -balanced natural transformation  $\hat{\eta} : \text{Hom}_{\mathcal{N}}(n, F(m)) \rightarrow \text{Hom}_{\mathcal{N}}(n, G(m))$  is mapped to the unique  $\mathcal{C}$ -module natural transformation  $\eta : F \rightarrow G$  with  $\hat{\eta}(f) = \eta(m) \circ f$  for all  $f \in \text{Hom}_{\mathcal{N}}(n, F(m))$ . For three module functors  $F, G, K : {}_{\mathcal{C}}\mathcal{M} \rightarrow {}_{\mathcal{C}}\mathcal{N}$ , the  $\mathcal{C}$ -module natural transformation  $F \rightarrow K$  corresponding to a composition  $\text{Hom}(n, F(m)) \rightarrow \text{Hom}(n, G(m)) \rightarrow \text{Hom}(n, K(m))$  of  $\mathcal{C}$ -balanced natural isomorphisms is equal to the composition of the corresponding  $\mathcal{C}$ -module natural transformations.

*Proof:* The usual Yoneda lemma shows that a transformation  $\hat{\eta} : \text{Hom}_{\mathcal{N}}(n, F(m)) \rightarrow \text{Hom}_{\mathcal{N}}(n, G(m))$  that is natural in both arguments can be identified with a natural transformation  $\eta : F \rightarrow G$ . Consider the following diagram.

$$\begin{array}{ccc} \text{Hom}(n, F(c \triangleright m)) & \xrightarrow{\eta(c \triangleright m)} & \text{Hom}(n, G(c \triangleright m)) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Hom}(n, c \triangleright F(m)) & \xrightarrow{c \triangleright \eta(m)} & \text{Hom}(n, c \triangleright G(m)) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Hom}(c^* \triangleright n, F(m)) & \xrightarrow{\eta(m)} & \text{Hom}(c^* \triangleright n, G(m)). \end{array} \quad (4.10)$$

The vertical isomorphisms provide the  $\mathcal{C}$ -balancing structure of the functor  $\text{Hom}(n, F(m))$ . It is easy to see that these isomorphisms satisfy the pentagon constraint. An analogous consideration holds for  $\text{Hom}(n, G(m))$ .

The lower rectangle in (4.10) commutes for any natural transformation  $\eta$ . The outer diagram commutes if and only if the upper rectangle commutes. The former commutes if and only if  $\eta$  is a  $\mathcal{C}$ -module natural transformation, while commutativity of the latter is precisely the condition on  $\eta$  to define a  $\mathcal{C}$ -balanced natural isomorphism  $\text{Hom}_{\mathcal{N}}(n, F(m)) \rightarrow \text{Hom}_{\mathcal{N}}(n, G(m))$ . The statement about the composition follows directly from the corresponding property of the Yoneda lemma.  $\square$

The next result shows that module traces are essentially unique. Consequently the existence of a module trace is a property of a module category over a pivotal fusion category rather than a structure on a module category.

**Proposition 4.4** *Let  $({}_e\mathcal{M}, \Theta)$  be an indecomposable module category over  $\mathcal{C}$  with module trace. For any other module trace  $\tilde{\Theta}$  on  ${}_e\mathcal{M}$  there is a  $z \in \mathbb{C}^\times$  such that  $\tilde{\Theta} = z \cdot \Theta$ .*

*Proof:* Let  $\Theta$  and  $\tilde{\Theta}$  be two module traces on  ${}_e\mathcal{M}$ . According to Theorem 4.2 they correspond to  $\mathcal{C}$ -balanced natural isomorphisms  $\eta, \tilde{\eta} : \text{Hom}(m, n) \rightarrow \text{Hom}(n, m)'$ , respectively. Hence the vertical composition  $\eta^{-1} \cdot \tilde{\eta} : \text{Hom}(m, n) \rightarrow \text{Hom}(m, n)$  of the natural isomorphisms is a  $\mathcal{C}$ -balanced natural isomorphism. According to Lemma 4.3 there is a unique  $\mathcal{C}$ -module natural isomorphism  $Z : \text{id}_{\mathcal{M}} \rightarrow \text{id}_{\mathcal{M}}$  such that

$$\eta^{-1} \cdot \tilde{\eta}(f) = Z(n) \circ f \quad \text{for all } f \in \text{Hom}(m, n). \quad (4.11)$$

Theorem 2.13 implies that there is a non-zero complex number  $z$  such that  $Z(f) = z \cdot f$  for all morphisms  $f$  in  $\mathcal{M}$ . Thus  $\tilde{\eta}(f) = z \cdot \eta(f)$  and so  $\tilde{\Theta} = z \cdot \Theta$ .  $\square$

## 4.2 The Double Adjoints of Module Functors

In this subsection we construct natural module isomorphisms between module functors of module categories with  $\mathcal{C}$ -module traces and their double adjoint module functors. These isomorphisms are compatible with the composition of functors and if the module category is indecomposable they define a pivotal structure for the dual fusion category. Recall that the left and right adjoint functors of a module functor  $F : {}_e\mathcal{M} \rightarrow {}_e\mathcal{N}$ ,  $F^l$  and  $F^r$ , have a canonical structure of module functors. Note that in our convention the left adjoint functor  $F^l$  is a right dual object to  $F$  in the tensor category of functors and natural transformations.

**Theorem 4.5** *Consider  ${}_e\mathcal{M}, {}_e\mathcal{N} \in \text{Mod}^\Theta(\mathcal{C})$ . For all module functors  $F : {}_e\mathcal{M} \rightarrow {}_e\mathcal{N}$  there is a canonical module natural isomorphism  $a_F : F \rightarrow F^{ll}$  to the double left adjoint module functor of  $F$ .*

- i) *The natural isomorphisms  $a_F$  are natural with respect to module natural transformations, i.e. for any module functor  $G : {}_e\mathcal{M} \rightarrow {}_e\mathcal{N}$  and any module natural transformation  $\rho : F \rightarrow G$ , the diagram*

$$\begin{array}{ccc} F & \xrightarrow{a_F} & F^{ll} \\ \rho \downarrow & & \downarrow \rho^{ll} \\ G & \xrightarrow{a_G} & G^{ll} \end{array} \quad (4.12)$$

*commutes.*

- ii) *For all module functors  $F : {}_e\mathcal{M} \rightarrow {}_e\mathcal{N}$  and  $K : {}_e\mathcal{N} \rightarrow {}_e\mathcal{E}$ ,*

$$a_{KF} = a_K \circ a_F : K \circ F \rightarrow (K \circ F)^{ll}. \quad (4.13)$$

*In particular, these isomorphisms equip the dual category  $\mathcal{C}_{\mathcal{M}}^* = \text{Fun}_e(\mathcal{M}, \mathcal{M})$  with a pivotal structure that is invariant under rescaling of the module trace of  ${}_e\mathcal{M}$ .*

*Proof:* According to Theorem 4.2 we can identify the module traces with  $\mathcal{C}$ -balanced natural isomorphisms  $\eta^M : \text{Hom}(m, \tilde{m}) \rightarrow \text{Hom}(\tilde{m}, m)'$  and  $\eta^N : \text{Hom}(n, \tilde{n}) \rightarrow \text{Hom}(\tilde{n}, n)'$ . Consider the following sequence of natural  $\mathcal{C}$ -balanced isomorphisms:

$$\begin{aligned} \text{Hom}_N(n, F(m)) &\simeq \text{Hom}_M(F^l(n), m) \xrightarrow{\eta^M} \text{Hom}_M(m, F^l(n))' \\ &\simeq \text{Hom}_N(F^{ll}(m), n)' \xrightarrow{(\eta^N)^{-1}} \text{Hom}_N(n, F^{ll}(m)). \end{aligned} \quad (4.14)$$

According to Lemma 4.3, the composition defines a  $\mathcal{C}$ -module natural isomorphism  $a_F : F \rightarrow F^{ll}$ .

For the first part we have to show that the diagram

$$\begin{array}{ccc}
 \text{Hom}(n, Fm) & \xrightarrow{\text{Hom}(n, \rho m)} & \text{Hom}(n, Gm) \\
 \downarrow \simeq & & \downarrow \simeq \\
 \text{Hom}(F^l n, m) & \xrightarrow{\text{Hom}(\rho^l n, m)} & \text{Hom}(G^l n, m) \\
 \downarrow \eta^{\mathcal{M}} & & \downarrow \eta^{\mathcal{M}} \\
 \text{Hom}(m, F^l n)' & \xrightarrow{\text{Hom}(m, \rho^l n)'} & \text{Hom}(m, G^l n)' \\
 \downarrow \simeq & & \downarrow \simeq \\
 \text{Hom}(F^{ll} m, n)' & \xrightarrow{\text{Hom}(\rho^{ll} m, n)'} & \text{Hom}(G^{ll} m, n)' \\
 \downarrow (\eta^{\mathcal{N}})^{-1} & & \downarrow (\eta^{\mathcal{N}})^{-1} \\
 \text{Hom}(n, F^{ll} m) & \xrightarrow{\text{Hom}(n, \rho^{ll} m)} & \text{Hom}(n, G^{ll} m)
 \end{array} \quad (4.15)$$

commutes. All subdiagrams commute either by naturality of  $\eta^{\mathcal{M}}$  and  $\eta^{\mathcal{N}}$ , by definition of the adjoint of  $\rho$ , or by definition of  $a_F$  and  $a_G$ . Hence the whole diagram commutes.

For the second part we identify  $(KF)^l = F^l K^l$ . It is enough to prove that the following diagram commutes:

$$\begin{array}{ccc}
 \text{Hom}(e, KFm) & & \text{Hom}(e, K^{ll} F^{ll} m) \\
 \downarrow \simeq & \searrow a_{KF} & \\
 \text{Hom}(F^l K^l e, m) & & \\
 \downarrow \eta^{\mathcal{M}} & & \nearrow (\eta^{\mathcal{E}})^{-1} \\
 \text{Hom}(m, F^l K^l e)' & \xrightarrow{\quad} & \text{Hom}(K^{ll} F^{ll} m, e)' \\
 \downarrow \simeq & \nearrow \simeq & \\
 \text{Hom}(F^{ll} m, K^l e)' & & \\
 \downarrow (\eta^{\mathcal{N}})^{-1} & & \nearrow a_{K F^{ll}} \\
 \text{Hom}(K^l e, F^{ll} m) & & \\
 \downarrow \simeq & \nearrow & \\
 \text{Hom}(e, K F^{ll} m) & & 
 \end{array} \quad (4.16)$$

The upper triangle and the lower subdiagram commute due to the definition of  $a_{KF}$  and  $a_K$ , respectively. It remains to show that the subdiagram on the left commutes. It is easy to see that this subdiagram can be rewritten as

$$\begin{array}{ccc}
 \text{Hom}(e, KFm) & \xrightarrow{\simeq} & \text{Hom}(K^l e, Fm) \\
 \downarrow \text{Hom}(e, K a_F m) & & \downarrow \text{Hom}(K^l e, a_F m) \\
 \text{Hom}(e, K F^{ll} m) & \xrightarrow{\simeq} & \text{Hom}(K^l e, F^{ll} m),
 \end{array} \quad (4.17)$$

The commutativity of the diagram (4.17) follows from the naturality of the adjunction and thus the second part is proven. From part *i)* and *ii)* it is clear that the isomorphisms  $a_F$

equip  $\mathcal{C}_{\mathcal{M}}^* = \text{Fun}_e(\mathcal{M}, \mathcal{M}) \ni F$  with a pivotal structure. As the construction of  $a_F$  involves the map  $\eta^{\mathcal{M}} : \text{Hom}_{\mathcal{M}}(m, \tilde{m}) \simeq \text{Hom}_{\mathcal{M}}(\tilde{m}, m)'$  composed with its inverse, a constant scale factor cancels out.  $\square$

**Corollary 4.6** *Let  ${}_e\mathcal{M} \in \text{Mod}^\Theta(\mathcal{C})$ . Consider  $\mathcal{M}$  as a  $\mathcal{C}_{\mathcal{M}}^*$ -left module category and equip  $\mathcal{C}_{\mathcal{M}}^*$  with the induced pivotal structure from Theorem 4.5. Then the  $\mathcal{C}$ -module trace on  $\mathcal{M}$  is also a  $\mathcal{C}_{\mathcal{M}}^*$ -module trace.*

*Proof:* See Section 2.2 for the structures of the category  $\mathcal{C}_{\mathcal{M}}^* = \text{Fun}_e(\mathcal{M}, \mathcal{M})$ . The action of a  $F \in \mathcal{C}_{\mathcal{M}}^*$  on  ${}_e\mathcal{M}$  is given by the application of the functor  $F$ . By Theorem 4.2 it is sufficient to show that the  $\mathcal{C}$ -balanced natural isomorphism  $\eta : \text{Hom}(m, \tilde{m}) \simeq \text{Hom}(\tilde{m}, m)'$  is also  $\mathcal{C}_{\mathcal{M}}^*$ -balanced. The induced pivotal structure provides a natural isomorphism  $a_F^r : F^r \rightarrow F^l$  for a functor  $F \in \mathcal{C}_{\mathcal{M}}^*$ . We have show that the diagram

$$\begin{array}{ccc}
 \text{Hom}(m, Fn) & \xrightarrow{\eta} & \text{Hom}(Fn, m)' \\
 \downarrow \simeq & & \downarrow \simeq \\
 & & \text{Hom}(n, F^r m)' \\
 & & \downarrow (a_F^r)^{-1} \\
 \text{Hom}(F^l m, n) & \xrightarrow{\eta} & \text{Hom}(n, F^l m)'
 \end{array} \tag{4.18}$$

commutes for all  $m, n \in \mathcal{M}$  and  $F \in \mathcal{C}_{\mathcal{M}}^*$ . The arrows pointing downwards are the  $\mathcal{C}$ -balancing natural isomorphism for  $\text{Hom}(m, n)$  and  $\text{Hom}(n, m)'$ , that are defined by the adjunction and the pivotal structure according to equation (4.5) and (4.6), respectively. The natural isomorphism  $a_F$  is defined by equation (4.14) in precisely such a way that the diagram commutes. Hence the statement follows.  $\square$

### 4.3 Conjugation of Pivotal Structures

When we restrict the considerations of the previous subsection to the case of  $\mathcal{C}$  as a left module category over itself, Theorem 4.5 provides a conjugation operation on the set of pivotal structures of a fusion category  $\mathcal{C}$ . We show how this conjugation can be obtained alternatively from a canonical natural monoidal isomorphism  $g : \text{id}_{\mathcal{C}} \rightarrow (.)^{****}$  that exists for all fusion categories  $\mathcal{C}$ . To avoid confusion we do not suppress the pivotal structures in the graphical calculations of this subsection.

**Theorem 4.7** *Let  $\mathcal{C}$  be a fusion category with pivotal structure  $a : \text{id}_{\mathcal{C}} \rightarrow (.)^{**}$ .*

- i) *There exists a pivotal structure  $\bar{a} : \text{id}_{\mathcal{C}} \rightarrow (.)^{**}$  for  $\mathcal{C}$  with  $(\bar{a}^{**x})^{-1} : x \rightarrow **x$  defined by*

$$\begin{array}{c}
 \begin{array}{c}
 *c \\
 \downarrow \\
 a*c \\
 \downarrow \\
 c^*
 \end{array}
 \begin{array}{c}
 \begin{array}{c}
 c \\
 \downarrow \\
 x \\
 \downarrow \\
 **x \\
 \downarrow \\
 c
 \end{array}
 \begin{array}{c}
 f \\
 \downarrow \\
 g
 \end{array}
 \end{array}
 (\bar{a}^{**x})^{-1} =
 \begin{array}{c}
 \begin{array}{c}
 *d \\
 \downarrow \\
 a*d \\
 \downarrow \\
 d^*
 \end{array}
 \begin{array}{c}
 \begin{array}{c}
 d \\
 \downarrow \\
 c \\
 \downarrow \\
 d
 \end{array}
 \begin{array}{c}
 **x \\
 \downarrow \\
 g \\
 \downarrow \\
 f
 \end{array}
 \end{array}
 **x, \tag{4.19}
 \end{array}$$

for all  $f \in \text{Hom}(c, d \otimes x)$  and  $g \in \text{Hom}(d \otimes **x, c)$ .

- ii) *The dimension of an object  $x$  with respect to the pivotal structure  $\bar{a}$  is equal to the dimension of  $*x$  with respect to  $a$ .*

iii)  $\bar{a} = a$  if and only if  $a$  is spherical.

iv)  $\bar{\bar{a}} = a$ .

*Proof:* It is well-known (see e.g. [7]) that  $\text{Fun}_e(\mathcal{C}, \mathcal{C})$  is canonically equivalent to  $\mathcal{C}^{\text{rev}}$  as a fusion category.  $\mathcal{C}^{\text{rev}}$  is the category  $\mathcal{C}$  with reversed tensor product. The module functors  ${}_e\mathcal{C} \rightarrow {}_e\mathcal{C}$  can be identified with the functors  $(.) \otimes x$  of right tensoring with objects  $x \in \mathcal{C}$ . The left adjoint functor to  $(.) \otimes x$  is  $(.) \otimes {}^*x$ . To show i) we introduce the following graphical notation for the isomorphism  $\eta^e : \text{Hom}(c, d) \rightarrow \text{Hom}(d, c)'$ :

(4.20)

Once the ellipse is replaced by a morphism  $h \in \text{Hom}(d, c)$ , the diagram represents the number  $\eta^e(f)(h)$ . The chain of isomorphisms (4.14) reads now in graphical terms:

(4.21)

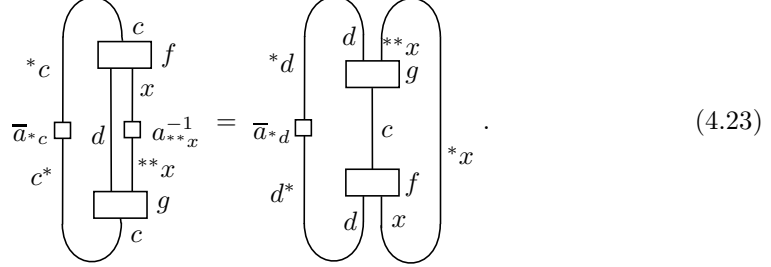
Inserting once more the definition of  $\eta^e$ , we conclude that equation (4.14) yields equation (4.19). Hence Theorem 4.5 implies the first part.

The second statement follows by restricting the first statement to the case  $d = {}^*x$ ,  $c = 1$ ,  $f = \text{coev}'_x$  and  $g = \text{ev}'_{*x}$ . Recall that we defined the dimensions in a pivotal category as the right trace of the identity morphism.

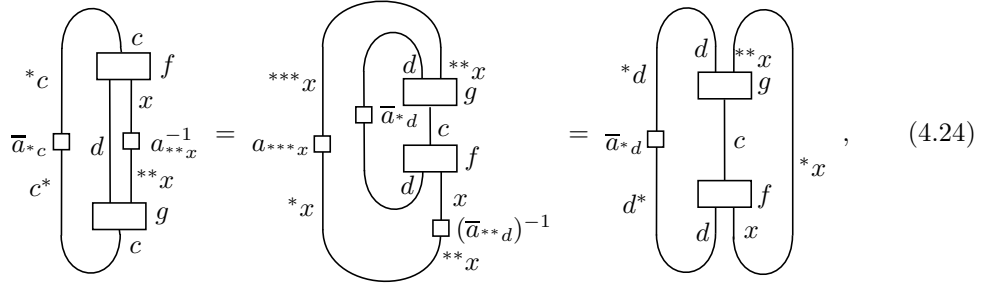
Now consider the case  $a = \bar{a}$ . The second part implies  $\dim^e(c) = \dim^e({}^*c)$  for all  $c \in \mathcal{C}$  and it follows that  $a$  is spherical (see [16]). Conversely, suppose that  $a$  is spherical. Then

(4.22)

where we used that  $a$  is spherical in the last step. So  $\bar{a} = a$  by equation (4.19). For part *iv)* we have to show that



With the symmetry of the right trace we calculate



where in the last step we used equation (4.19) with the morphism  $g$  in (4.19) set to  $\text{id}_{**x}$ . This proves the theorem.  $\square$

We call the pivotal structure  $\bar{a}$  the conjugate pivotal structure of  $a$ . In the example of  $G$ -graded vector spaces, see 3.13, where a pivotal structure is a group homomorphism  $\kappa : G \rightarrow \mathbb{C}$ , the conjugate pivotal structure is indeed given by the complex conjugate of  $\kappa$ .

It is instructive to consider the existence of conjugate pivotal structures also from another perspective. In [7] it is shown that for every fusion category there exists a monoidal natural isomorphism  $g : \text{id} \rightarrow (.)^{***}$ . We provide a simple description of such an isomorphism using dual  $\text{Hom}$ -spaces and show that the conjugate of a pivotal structure can be constructed with this isomorphism. We remark that in [3] another graphical proof of the existence of such a natural isomorphism  $g$  is given with a different approach to pivotal structures.

**Proposition 4.8** *Let  $\mathcal{C}$  be a fusion category.*

*i) The map*

$$\phi_c : \text{Hom}(c, 1) \rightarrow \text{Hom}(1, c)', \quad \phi(f)(h) = h \circ f \in \mathbb{C} \quad (4.25)$$

*for  $c \in \mathcal{C}$ ,  $f \in \text{Hom}(1, c)$  and  $h \in \text{Hom}(c, 1)$  is a natural isomorphism.*

*ii) The following chain of isomorphisms*

$$\begin{aligned} \text{Hom}(x, **c) &\simeq \text{Hom}(*c \otimes x, 1) \xrightarrow{\phi} \text{Hom}(1, *c \otimes x)' \simeq \text{Hom}(c, x)' \\ &\simeq \text{Hom}(1, x \otimes c^*)' \xrightarrow{\phi^{-1}} \text{Hom}(x \otimes c^*, 1) \simeq \text{Hom}(x, c^{**}) \end{aligned} \quad (4.26)$$

*is natural in  $c, x \in \mathcal{C}$  and defines a monoidal natural isomorphism  $g_c : **c \rightarrow c^{**}$ .*

iii)  $g_c : **c \rightarrow c^{**}$  is defined uniquely by the requirement that for all  $f \in \text{Hom}(x, **c)$  and  $h \in \text{Hom}(c, x)$ :

$$\begin{array}{c}
 c \\
 | \\
 \boxed{\phantom{x}} \quad h \\
 | \\
 x \\
 | \\
 \boxed{\phantom{x}} \quad f \\
 | \\
 **c
 \end{array}
 = 
 \begin{array}{c}
 c \\
 | \\
 \boxed{\phantom{x}} \quad h \\
 | \\
 x \\
 | \\
 \boxed{\phantom{x}} \quad f \\
 | \\
 **c \\
 \boxed{\phantom{x}} \quad g_c \\
 | \\
 c^{**}
 \end{array}
 . \tag{4.27}$$

*Proof:* The naturality of  $\phi$  in part *i*) is clear.  $\phi$  is an isomorphism due to the semisimplicity of  $\mathbb{C}$ . For part *ii*), the naturality of the isomorphisms in  $x$  and  $c$  is a consequence of part *i*) and the naturality of the duality. Hence the isomorphism  $g_c$  is well-defined by the Yoneda lemma. We introduce the graphical notation

$$\begin{array}{c} \text{---} \\ | \\ c \\ | \\ \text{---} \\ f \end{array} \quad (4.28)$$

for  $\phi_c(f) \in \text{Hom}(1, c)'$ . If the unlabelled ellipse is replaced by an morphism  $h \in \text{Hom}(1, c)$ , this expression represents the number  $\phi_c(f)(h)$ . Now the chain of isomorphisms (4.26) reads in graphical terms

$$\begin{array}{c}
\begin{array}{c} x \\ \boxed{\phantom{x}} f \\ \text{**}_c \end{array} \mapsto \begin{array}{c} x \\ \boxed{\phantom{x}} f \\ \text{**}_c \end{array} \xrightarrow{\phi} \begin{array}{c} x \\ \boxed{\phantom{x}} f \\ \text{**}_c \end{array} \mapsto \begin{array}{c} x \\ \boxed{\phantom{x}} f \\ \text{**}_c \end{array} \\
\mapsto \begin{array}{c} x \\ \boxed{\phantom{x}} f \\ \text{**}_c \end{array} \xrightarrow{\phi^{-1}} \begin{array}{c} x \\ \boxed{\phantom{x}} \tilde{f} \\ \text{**}_c \end{array} \mapsto \begin{array}{c} x \\ \boxed{\phantom{x}} \tilde{f} \\ \text{**}_c \end{array} = \begin{array}{c} x \\ \boxed{\phantom{x}} f \\ \text{**}_c \\ \boxed{\phantom{x}} g_c \\ \text{**}_c \end{array},
\end{array}
\tag{4.29}$$

where  $\tilde{f}$  is defined by

$$\begin{array}{c} \text{O} \\ | \\ \text{---} x \text{---} c^* \\ | \\ \text{---} f \text{---} \end{array} = \begin{array}{c} \text{O} \\ | \\ \text{---} x \text{---} c^* \\ | \\ \text{---} f \text{---} \end{array} \begin{array}{c} \text{---} c \\ | \\ \text{---} c^* \\ | \\ \text{---} c \\ | \\ \text{---} c \end{array} . \quad (4.30)$$



Applying the rigidity of  $\mathcal{C}$  it follows that

$$(4.31)$$

Applying once more the rigidity of  $\mathcal{C}$ , equation (4.31) implies expression (4.26). The compatibility of  $g$  with the monoidal structure is easy to see from the graphical expression (4.27).  $\square$

It seems plausible that this monoidal natural isomorphism  $\text{id} \rightarrow (.)^{***}$  coincides with the isomorphisms defined in [7] and [3] but it remains to show that they are indeed equal. The following proposition clarifies the relation between  $g$  and the conjugate of a pivotal structure.

**Proposition 4.9** *Let  $\mathcal{C}$  be a fusion category with pivotal structure  $a : \text{id} \rightarrow (.)^{**}$ .*

- i)  $a$  and its conjugate  $\bar{a}$  combine to  $g$ , i.e.  $\bar{a}_c \cdot a_{**c} = a_c \cdot \bar{a}_{**c} = g_c : **c \rightarrow c^{**}$ .*
- ii)  $a$  is spherical if and only if  $a_c \cdot a_{**c} = g_c$ .*

*Proof:* For all  $f : c \rightarrow **c$ ,

$$(4.32)$$

by equation (4.19). This implies  $a_c \cdot \bar{a}_{**c} = g_c$  with condition (4.27). The other equation follows directly from the naturality of  $a$ . For the second part note that the first part implies  $\bar{a}_c = g_c \cdot a_{**c}^{-1}$ . Now the statement follows directly from Theorem 4.7, *iii*). The statement can also be derived directly from the graphical expression (4.27).  $\square$

## 5 The Existence Problem as an Eigenvalue Equation

The aim of this section is to formulate the existence of a module trace as an eigenvalue problem. In particular this allows one to show that all module categories over pseudo-unitary fusion categories equipped with the canonical spherical structure admit a module trace. We restrict attention to module categories with finitely many isomorphism classes of simple objects in this section.

### 5.1 The Dimension Matrix of a Module Category

We show how a trace on a semisimple category is characterised by the dimensions of simple objects using the trace in **Vect**. For a module trace on a module category over  $\mathcal{C}$  we derive an analogous formula with the trace in **Vect** replaced by the right trace in  $\mathcal{C}$ . As a consequence we obtain that the existence of a module trace on  ${}_c\mathcal{M}$  implies

$\dim^{\mathcal{C}}({}_e\langle m, m \rangle) > 0$  for all simple  $m \in \mathcal{M}$ . In the last part we apply the considerations to spherical fusion categories and show that a pivotal structure for  $\mathcal{C}$  is spherical if and only if there is a module category  ${}_e\mathcal{M}$  over  $\mathcal{C}$  with a module trace such that all dimensions in  $\mathcal{M}$  are real.

Consider general traces on a semisimple category  $\mathcal{M}$  with a finite set of representatives  $m_i$ ,  $i \in I$  for the isomorphism classes of simple objects. The following lemma is well-known, see e.g. [20, Lemma II.4.2.3].

**Lemma 5.1** *A collection of linear maps  $\Theta_m : \text{End}_{\mathcal{M}}(m) \rightarrow \mathbb{C}$  that satisfies the symmetry property of Definition 3.7 i) is non-degenerate and hence a trace on  $\mathcal{M}$  if and only if  $\Theta(\text{id}_{m_i}) \neq 0$  for all  $i \in I$ .*

**Proposition 5.2** *For every trace  $\Theta$  on  $\mathcal{M}$ ,  $(\dim^{\Theta}(m_i))_{i \in I}$  is an  $|I|$ -tuple of non-zero numbers. Conversely, given such a tuple  $d_i \in \mathbb{C}^{\times}$ ,  $i \in I$ ,*

$$\Theta_m(f) = \sum_{i \in I} \text{tr}(\text{Hom}(m_i, f)) d_i, \quad (5.1)$$

for  $f \in \text{Hom}(m, m)$  defines a trace on  $\mathcal{M}$ . Here  $\text{tr}(\text{Hom}(m_i, f))$  denotes the usual trace on  $\text{Vect}$  of the linear map  $\text{Hom}(m_i, f) : \text{Hom}(m_i, m) \rightarrow \text{Hom}(m_i, m)$ .

These two maps yield a bijection between the set of traces on  $\mathcal{M}$  and the set of  $|I|$ -tuples of non-zero numbers.

*Proof:* Suppose that  $\mathcal{M}$  is equipped with a trace  $\Theta$ . Then  $d_i = \dim^{\Theta}(m_i) \neq 0$  due to Lemma 5.1. We have to show that for all  $f \in \text{End}(m)$  formula (5.1) holds. The semisimplicity of  $\mathcal{M}$  ensures that the functor

$$\mathcal{M} \ni m \mapsto \oplus_i \text{Hom}_{\mathcal{M}}(m_i, m) \otimes_{\mathbb{C}} m_i \quad (5.2)$$

is naturally isomorphic to the identity functor on  $\mathcal{M}$ . This implies

$$\begin{aligned} \Theta_m(f) &= \Theta_{\oplus_i \text{Hom}(m_i, m) \otimes_{\mathbb{C}} m_i}(\oplus_i \text{Hom}(m_i, f) \otimes_{\mathbb{C}} m_i) \\ &= \sum_{i \in I} \Theta_{\text{Hom}(m_i, m) \otimes_{\mathbb{C}} m_i}(\text{Hom}(m_i, f) \otimes_{\mathbb{C}} m_i) \\ &= \sum_{i \in I} \text{tr}(\text{Hom}(m_i, f)) d_i, \end{aligned} \quad (5.3)$$

where we used Lemma 3.12 in the last step.

For the converse we have to show that given a set of non-zero  $d_i \in \mathbb{C}$  for  $i \in I$ , formula (5.1) defines a trace on  $\mathcal{M}$ . The symmetry follows directly from the cyclic property of  $\text{tr}$ . The non-degeneracy follows from Lemma 5.1.  $\square$

Now we discuss  $\mathcal{C}$ -module traces. First we need a technical result. Choose representatives  $c_u$ ,  $u \in U$  for the isomorphism classes of simple objects of  $\mathcal{C}$ . See e.g. [11] for a review of the definition of the Deligne product  $\boxtimes$  of additive categories.

**Lemma 5.3** *The following functors  $\mathcal{M} \rightarrow \mathcal{C} \boxtimes \mathcal{M}$  are naturally isomorphic.*

$$\begin{aligned} m &\mapsto \oplus_{u \in U} c_u \boxtimes {}^*c_u \triangleright m, \quad \text{and} \\ m &\mapsto \oplus_{i \in I} {}_e\langle m_i, m \rangle \boxtimes m_i. \end{aligned} \quad (5.4)$$

*Proof:* The objects  $\oplus_{u \in U} c_u \boxtimes {}^*c_u \in \mathcal{C} \boxtimes \mathcal{C}$  and  $\oplus_{i \in I} m_i \boxtimes m_i \in \mathcal{M}^{\text{op}} \boxtimes \mathcal{M}$  are independent of the choice of representatives of simple objects in the sense that the objects obtained from any two choices of simple objects are canonically isomorphic, see [1, Sec. 2.4]. This

shows that the two maps yield well-defined functors. Now let  $c \boxtimes \tilde{m} \in \mathcal{C} \boxtimes \mathcal{M}$ . Using the semisimplicity of  $\mathcal{C}$  and  $\mathcal{M}$  we obtain the following chain of natural isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C} \boxtimes \mathcal{M}}(c \boxtimes \tilde{m}, \oplus_u c_u \boxtimes {}^*c_u \triangleright m) &\simeq \oplus_u \mathrm{Hom}_{\mathcal{C}}(c, c_u) \otimes \mathrm{Hom}_{\mathcal{M}}(\tilde{m}, {}^*c_u \triangleright m) \\
&\simeq \oplus_u \mathrm{Hom}_{\mathcal{C}}(c, c_u) \otimes \mathrm{Hom}_{\mathcal{C}}(c_u, {}_e\langle \tilde{m}, m \rangle) \\
&\simeq \mathrm{Hom}_{\mathcal{C}}(c, {}_e\langle \tilde{m}, m \rangle) \simeq \mathrm{Hom}(c \triangleright \tilde{m}, m) \\
&\simeq \mathrm{Hom}_{\mathcal{M}}(\tilde{m}, {}^*c \triangleright m) \\
&\simeq \oplus_i \mathrm{Hom}_{\mathcal{M}}(m_i, {}^*c \triangleright m) \otimes \mathrm{Hom}_{\mathcal{M}}(\tilde{m}, m_i) \\
&\simeq \oplus_i \mathrm{Hom}_{\mathcal{C}}(c, {}_e\langle m_i, m \rangle) \otimes \mathrm{Hom}_{\mathcal{M}}(\tilde{m}, m_i) \\
&\simeq \mathrm{Hom}_{\mathcal{C} \boxtimes \mathcal{M}}(c \boxtimes \tilde{m}, \oplus_i {}_e\langle m_i, m \rangle \boxtimes m_i).
\end{aligned} \tag{5.5}$$

Now apply the Yoneda lemma to obtain a natural isomorphism between the two functors.  $\square$

The following result provides an alternative characterisation of module traces. Recall from [7] that for a pivotal fusion category  $\dim(\mathcal{C}) = \sum_{u \in U} |\dim^{\mathcal{C}}(c_u)|^2 \neq 0$ .

**Proposition 5.4** *Let  ${}_e\mathcal{M}$  be a  $\mathcal{C}$ -module category. If  $\Theta$  is a  $\mathcal{C}$ -module trace on  $\mathcal{M}$ , the dimension vector  $d_i = \dim^{\Theta}(m_i)$  for  $i \in I$  consists of non-zero numbers  $d_i$  and is a (right) eigenvector of the matrix  $(Q)_{ij} = \dim^{\mathcal{C}}({}_e\langle m_j, m_i \rangle)$  with eigenvalue  $\dim(\mathcal{C})$ . If a tuple of non-zero numbers  $d_i$  for  $i \in I$  is an eigenvector of  $(Q)_{ij}$  with eigenvalue  $\dim(\mathcal{C})$ , then the collection of linear maps*

$$\Theta_m(f) = \frac{1}{\dim(\mathcal{C})} \sum_{i \in I} \mathrm{tr}^R({}_e\langle m_i, f \rangle) d_i, \tag{5.6}$$

for  $f \in \mathrm{End}(m)$  and  $m \in \mathcal{M}$  defines a  $\mathcal{C}$ -module trace on  $\mathcal{M}$ . These two maps are mutually inverse.

*Proof:* Let  $\Theta$  be  $\mathcal{C}$ -module trace on  $\mathcal{M}$ . Lemma 5.3 implies that the object  $\oplus_u (c_u \otimes {}^*c_u) \triangleright m$  is isomorphic to  $\oplus_i {}_e\langle m_i, m \rangle \triangleright m_i$  in  $\mathcal{M}$ . Hence,

$$\begin{aligned}
\dim(\mathcal{C}) \cdot d_k &= \dim^{\mathcal{C}}(\oplus_u (c_u \otimes {}^*c_u)) \cdot \dim^{\Theta}(m_k) \\
&= \dim^{\Theta}(\oplus_u (c_u \otimes {}^*c_u) \triangleright m_k) = \dim^{\Theta}(\oplus_i {}_e\langle m_i, m_k \rangle \triangleright m_i) \\
&= \sum_{i \in I} \dim^{\mathcal{C}}({}_e\langle m_i, m_k \rangle) d_i.
\end{aligned} \tag{5.7}$$

In the sequel we will refer to the matrix  $Q = (Q_{ij})$  as the dimension matrix and to the vector  $d = (d_i)$  as the dimension vector. Equation (5.7) shows that the dimension vector is a right eigenvalue of the dimension matrix with eigenvalue  $\dim(\mathcal{C})$ . As another consequence of Lemma 5.3 we obtain the identity

$$\Theta(\oplus_u (c_u \otimes {}^*c_u) \triangleright f) = \Theta(\oplus_i {}_e\langle m_i, f \rangle \triangleright m_i), \tag{5.8}$$

for all  $f \in \mathrm{End}(m)$ . This implies formula (5.6) with  $d_i = \dim^{\Theta}(m_i)$ .

Now suppose we are given an eigenvector  $d$  of the dimension matrix with eigenvalue  $\dim(\mathcal{C})$  whose components do not vanish. Then define a linear map  $\Theta_m : \mathrm{End}(m) \rightarrow \mathbb{C}$  by the formula (5.6). The symmetry of  $\Theta$  follows from the cyclic property of the right trace  $\mathrm{tr}^R$  of  $\mathcal{C}$ . Since  $\Theta_{m_i}(\mathrm{id}_{m_i}) = \frac{1}{\dim(\mathcal{C})} \sum_j Q_{ij} d_j = d_i \neq 0$ , we conclude with Lemma 5.1 that  $\Theta$  is a trace on  $\mathcal{M}$ . For the  $\mathcal{C}$ -compatibility we have to show that for all  $f \in \mathrm{End}(c \triangleright m)$ ,

$$\sum_{i \in I} \mathrm{tr}^R({}_e\langle m_i, f \rangle) d_i = \sum_{i \in I} \mathrm{tr}^R({}_e\langle m_i, \mathrm{tr}^{\mathcal{C}}(f) \rangle) d_i. \tag{5.9}$$

Since  ${}_e\langle m_i, \cdot \rangle : {}_e\mathcal{M} \rightarrow {}_e\mathcal{C}$  is a module functor, Lemma 3.6 implies that  $\mathrm{tr}^{\mathcal{C}}({}_e\langle m_i, f \rangle) = {}_e\langle m_i, \mathrm{tr}^{\mathcal{C}}(f) \rangle$ . Now the statement follows from  $\mathrm{tr}^R({}_e\langle m_i, f \rangle) = \mathrm{tr}^R \circ \mathrm{tr}^{\mathcal{C}}({}_e\langle m_i, f \rangle)$ .  $\square$

Note that formula (5.6) is a generalisation of formula (5.1).

**Remark 5.5** The proof of Proposition 5.4 shows that for any set of numbers  $d_i$ ,  $i \in I$ , formula (5.6) defines a collection of linear maps that satisfy the symmetry and  $\mathcal{C}$ -compatibility condition of Definition 3.7. The non-degeneracy condition is fulfilled if and only if  $\sum_i Q_{ij}d_j \neq 0$  for all  $i \in I$ .

Next we discuss some properties of the dimension matrix for a module category  ${}_e\mathcal{M}$  that not necessarily possesses a module trace. Let

$$M_{u,i}^j = \dim_{\mathbb{C}}(\text{Hom}_{\mathcal{M}}(c_u \triangleright m_i, m_j)) \quad (5.10)$$

be the multiplicity matrix of the action of  $c_u \in \mathcal{C}$  on  $\mathcal{M}$ .

**Proposition 5.6** *Let  ${}_e\mathcal{M}$  be a  $\mathcal{C}$ -module category. The dimension matrix  $Q$  satisfies:*

- i)  $Q_{ij} = \sum_{u \in U} \dim^{\mathcal{C}}(c_u) M_{u,j}^i$ .
- ii)  $Q^2 = \dim(\mathcal{C}) \cdot Q$ .
- iii)  $Q$  is hermitian.

*Proof:* The multiplicity of each object  $c_u$  in  ${}_e\langle m_j, m_i \rangle$  is

$$\dim_{\mathbb{C}}(\text{Hom}(c_u, {}_e\langle m_j, m_i \rangle)) = \dim_{\mathbb{C}}(\text{Hom}(c_u \triangleright m_j, m_i)) = M_{u,j}^i. \quad (5.11)$$

This shows part i). For the second claim we first compute

$$\begin{aligned} \oplus_{j \in I} {}_e\langle m_j, m_i \rangle \otimes {}_e\langle m_k, m_j \rangle &= \oplus_{j \in j} {}_e\langle m_k, {}_e\langle m_j, m_i \rangle \triangleright m_j \rangle \\ &\simeq {}_e\langle m_k, \oplus_{u \in U} (c_u \otimes {}^*c_u) \triangleright m_j \rangle \\ &\simeq \oplus_{u \in U} (c_u \otimes {}^*c_u) \otimes {}_e\langle m_k, m_j \rangle, \end{aligned} \quad (5.12)$$

where we used Lemma 5.3 in the second step. Now the statement follows after applying  $\dim^{\Theta}$  to both sides of this equation. For the third statement we show that the objects  ${}_e\langle m_i, m_j \rangle$  and  ${}_e\langle m_j, m_i \rangle^*$  are isomorphic in  $\mathcal{C}$ . We compute the multiplicity of a  $c \in \mathcal{C}$  in  ${}_e\langle m_j, m_i \rangle^*$  by using that  $\text{Hom}(m, n) \simeq \text{Hom}(n, m)$  as vector spaces. The following isomorphisms are isomorphisms of vector spaces:

$$\begin{aligned} \text{Hom}(c, {}_e\langle m_j, m_i \rangle^*) &\simeq \text{Hom}({}_e\langle m_j, m_i \rangle, {}^*c) \simeq \text{Hom}({}^*c, {}_e\langle m_j, m_i \rangle) \\ &\simeq \text{Hom}({}^*c \triangleright m_j, m_i) \simeq \text{Hom}(m_i, {}^*c \triangleright m_j) \\ &\simeq \text{Hom}(c \triangleright m_i, m_j) = \text{Hom}(c, {}_e\langle m_i, m_j \rangle). \end{aligned} \quad (5.13)$$

As the multiplicities of all simple objects agree, we conclude that there exists an isomorphism  ${}_e\langle m_i, m_j \rangle \rightarrow {}_e\langle m_j, m_i \rangle^*$  in  $\mathcal{C}$ . With  $\dim^{\mathcal{C}}(c^*) = \dim^{\mathcal{C}}(c)$  for all objects  $c \in \mathcal{C}$  from [7, Proposition 2.9], it follows that

$$\dim^{\Theta}({}_e\langle m_i, m_j \rangle) = \dim^{\Theta}({}_e\langle m_j, m_i \rangle^*) = \overline{\dim^{\Theta}({}_e\langle m_j, m_i \rangle)}. \quad (5.14)$$

□

**Proposition 5.7** *A module category  ${}_e\mathcal{M}$  has a module trace if and only if the dimension matrix  $Q$  is of rank 1 with only non-zero entries. In particular is  $\dim^{\mathcal{C}}({}_e\langle m, m \rangle) > 0$  for all simple objects  $m \in \mathcal{M}$ .*

*Proof:* It follows directly from Proposition 5.6, that the only possible (right and left) eigenvalues of  $Q$  are  $\dim(\mathcal{C})$  and 0. Suppose  $\mathcal{M}$  has a module trace and  $d$  is the corresponding eigenvector of  $Q$  with all entries non-zero. Let  $\tilde{d}$  be an eigenvector of  $Q$  with eigenvalue  $\dim(\mathcal{C})$ . There always exists a linear combination  $d + \lambda \tilde{d}$  with all entries non-zero. Hence  $\tilde{d}$  must be proportional to  $d$ . This shows that  $Q$  has rank 1.

Now suppose  $Q_{ij} = d_j \overline{d_i}$  with non-zero numbers  $d_i$ . Then  $\sum_i \overline{d_i} d_i = \dim(\mathcal{C})$  by Proposition 5.6 ii). Hence  $d_i$  yields a module trace. This proves also the last statement since  $\dim^{\mathcal{C}}({}_e\langle m_i, m_i \rangle) = d_i \overline{d_i}$ .  $\square$

As an example we discuss pseudo-unitary fusion categories. Recall from [7] the definition of the Frobenius-Perron dimensions of objects in a fusion category. A pseudo-unitary fusion category possesses a canonical spherical structure such that the dimension of all objects are equal to the Frobenius-Perron dimensions. The following statement is a direct consequence of the results in [7] together with Proposition 5.4.

**Proposition 5.8** *Let  $\mathcal{C}$  be a pseudo-unitary fusion category. The canonical spherical structure of  $\mathcal{C}$  is flexible.*

*Proof:* Let  ${}_e\mathcal{M}$  be a module category over  $\mathcal{C}$ . We have to show that there exists a module trace for  $\mathcal{M}$ . There exists a Frobenius-Perron eigenvector  $(d_i)_{i \in I}$  of  $\mathcal{M}$ , that is defined by  $d_j > 0$  for all  $j \in I$  and:

$$\sum_{u \in U} M_{u,i}^j d_j = \dim^{\mathcal{C}}(c_u) d_i, \quad (5.15)$$

see [7]. If we multiply this equation with  $\dim^{\mathcal{C}}(c_u)$ , sum over  $u \in U$  and use that the pivotal structure is spherical, we see that  $(d_i)$  is an eigenvector of  $Q_{ij}$  with eigenvalue  $\dim(\mathcal{C})$  and hence defines a module trace according to Proposition 5.4.  $\square$

## 5.2 Module Traces on Module Categories over Spherical Fusion Categories

Next we discuss the relation of module traces and spherical structure.

**Proposition 5.9** *Let  $\mathcal{C}$  be spherical,  $\mathcal{M}$  a left  $\mathcal{C}$ -module category with module trace  $\Theta$ . There exists a  $z \in \mathbb{C}$  such that the dimensions of objects in  $\mathcal{M}$  with respect to the module trace  $z \cdot \Theta$  are real.*

*Proof:* If  $\mathcal{C}$  is spherical all dimensions of  $\mathcal{C}$  are real. Hence  $Q$  is a real symmetric matrix which can be diagonalised by a real matrix. It follows that the entries of all eigenvectors of  $Q$  are real.  $\square$

The next result provides a criterion to determine whether a given pivotal structure is spherical.

**Proposition 5.10** *Let  ${}_e\mathcal{M}$  be a module category with module trace  $\Theta$ .*

- i) *The dimension vector  $d_i = \dim^{\Theta}(m_i)$  is a left eigenvector of the dimension matrix with eigenvalue  $C = \sum_{u \in K} \dim(c_u)^2$ , i.e.  $\sum_j d_j Q_{ji} = C \cdot d_i$ .*
- ii) *The number  $C = \sum_{u \in K} \dim(c_u)^2$  is equal to  $\dim(\mathcal{C})$  if and only if the pivotal structure is spherical, and it is equal to 0 otherwise.*
- iii) *A pivotal structure for  $\mathcal{C}$  is spherical if and only if there exists a module category  ${}_e\mathcal{M}$  with module trace such that all dimensions of objects in  $\mathcal{M}$  are real.*
- iv) *Let  $\mathcal{C}$  be spherical and assume that  ${}_e\mathcal{M}$  has a module trace. Then the induced pivotal structure for the dual category  $\mathcal{C}_{\mathcal{M}}^*$  from Theorem 4.5 is spherical.*

*Proof:* The  $\mathcal{C}$ -compatibility of  $\Theta$  implies

$$\sum_i M_{u,j}^i d_i = \dim^{\Theta}(c_u \triangleright m_j) = \dim^{\mathcal{C}}(c_u) \cdot d_j. \quad (5.16)$$

Multiplying this equation with  $\dim^{\mathcal{C}}(c_u)$  and summing over  $u \in U$  yields:

$$\begin{aligned} C \cdot d_j &= \sum_{i \in I, u \in U} \dim^{\mathcal{C}}(c_u) M_{u,j}^i d_i \\ &= \sum_{i \in I} \dim^{\mathcal{C}}(\langle m_j, m_i \rangle) d_i, \end{aligned} \tag{5.17}$$

where we used Proposition 5.6, *i*). This proves the first statement. For the module category  ${}_e\mathcal{C}$ , equation (5.17) implies  $C \cdot \dim^{\mathcal{C}}(c_j) = C \cdot \dim^{\mathcal{C}}(c_j^*)$ . It is shown in [16] that  $\mathcal{C}$  is spherical if and only if  $\dim^{\mathcal{C}}(c_u^*) = \dim^{\mathcal{C}}(c_u)$ . Hence the second statement follows. To prove part *iii*), let  ${}_e\mathcal{M}$  be a module category with module trace  $\Theta$  and  $d_i = \dim^{\Theta}(m_i) \in \mathbb{R}$  for all simple  $m_i \in \mathcal{M}$ . According to Proposition 5.7 we can assume, using the freedom to rescale  $\Theta$ , that  $\sum_i d_i^2 = \dim(\mathcal{C})$  and therefore  $Q_{ij} = d_i d_j$ . From part *i*) it follows that  $C = \dim(\mathcal{C})$  and hence part *ii*) implies that the pivotal structure is spherical. The converse is clear since the module category  ${}_e\mathcal{C}$  has real dimensions if  $\mathcal{C}$  is spherical. The last statement is a consequence of part *iii*) together with Proposition 5.9 and Corollary 4.6.  $\square$

**Remark 5.11** It is shown in [16, Theorem 5.16] by different methods that an indecomposable module category  ${}_e\mathcal{M}$  over a spherical category  $\mathcal{C}$  provides a spherical structure for the category  $\text{Fun}_e(\mathcal{M}, \mathcal{M})$ . The relation to our construction remains to be investigated.

## 6 Frobenius Algebras

In this section we show that module traces are directly related to Frobenius algebras. This is done by exploring the graphical calculus for module categories with module traces and constructing a natural isomorphism  $\beta$  that is the reflected analogue of the  $\alpha$  in Subsection 3.1. This operation equips the inner hom objects with the structure of a Frobenius algebra. We also prove the converse, namely that the module category formed by the modules over a special haploid Frobenius algebra has a module trace.

To emphasise the role of the  $\mathcal{C}$ -compatibility of a module trace we first discuss traces on a module category  ${}_e\mathcal{M}$ . We saw in Section 4.1 that a module category with a trace that is not necessarily  $\mathcal{C}$ -compatible equips  ${}_e\mathcal{M}$  with a natural isomorphism  $\eta^{\mathcal{M}} : \text{Hom}_{\mathcal{M}}(m, n) \rightarrow \text{Hom}_{\mathcal{M}}(n, m)'$ . Recall that the pivotal structure of  $\mathcal{C}$  also yields a trace and a natural isomorphism  $\eta^{\mathcal{C}} : \text{Hom}_{\mathcal{C}}(c, d) \rightarrow \text{Hom}_{\mathcal{C}}(d, c)'$ , see equation (4.8).

**Proposition 6.1** *Let  ${}_e\mathcal{M}$  be a  $\mathcal{C}$ -module category equipped with a trace  $\Theta$ . Then there exists a natural isomorphism*

$$\beta : \text{Hom}(n, c \triangleright m) \rightarrow \text{Hom}(\langle m, n \rangle, c), \tag{6.1}$$

which is specified uniquely by the requirement

$$\text{tr}^R(\beta(f) \circ \alpha(g)) = \Theta_{c \triangleright m}(f \circ g), \tag{6.2}$$

for all  $g \in \text{Hom}(c \triangleright m, n)$  and with  $f \in \text{Hom}(n, c \triangleright m)$ .

*Proof:* Condition (6.2) is equivalent to defining  $\beta$  as the following composition of natural isomorphisms:

$$\begin{aligned} \text{Hom}(n, c \triangleright m) &\xrightarrow{\eta^{\mathcal{M}}} \text{Hom}(c \triangleright m, n)' \\ &\simeq \text{Hom}(c, \langle m, n \rangle)' \xrightarrow{(\eta^{\mathcal{C}})^{-1}} \text{Hom}(\langle m, n \rangle, c). \end{aligned} \tag{6.3}$$

This follows directly from the identity  $\eta^{\mathcal{M}}(a)(b) = \Theta_n(a \circ b)$  for  $a \in \mathbf{Hom}(m, n)$  and  $b \in \mathbf{Hom}(n, m)$ .  $\square$

The graphical representation of  $\beta$  is

$$\beta \cong \begin{array}{c} \text{---} n \\ | \\ \boxed{\phantom{0}} \\ | \\ c \text{---} m \end{array} \rightarrow \begin{array}{c} \text{---} n \text{---} m \\ | \\ \boxed{\phantom{0}} \\ | \\ c \end{array}, \quad (6.4)$$

i.e.  $\beta$  allows one to flip strings representing objects in the module category upwards. Equation (6.2) reads in graphical terms:

$$\begin{array}{c} \text{---} c \\ | \\ \boxed{\phantom{0}} \alpha(g) \\ | \\ n \text{---} m \\ | \\ \boxed{\phantom{0}} \beta(f) \\ | \\ \text{---} c \end{array} = \begin{array}{c} \text{---} c \text{---} m \\ | \\ \boxed{\phantom{0}} g \\ | \\ n \text{---} m \\ | \\ \boxed{\phantom{0}} f \\ | \\ c \end{array}. \quad (6.5)$$

The properties of  $\beta$  are analogous to the properties of  $\alpha$  from Subsection 3.1 provided that  $\Theta$  is a module trace.

**Proposition 6.2** *Let  ${}_e\mathcal{M}$  be a module category with module trace. Then the map  $\beta : \mathbf{Hom}(n, c \triangleright m) \rightarrow \mathbf{Hom}({}_e\langle m, n \rangle, c)$  is compatible with the module structure: For all morphisms  $\gamma : x \rightarrow y$  in  $\mathcal{C}$  and all  $f \in \mathbf{Hom}(n, c \triangleright m)$ ,*

$$\beta(\gamma \triangleright f) = \gamma \otimes \beta(f). \quad (6.6)$$

*Proof:* By Proposition 6.1,  $\beta(\gamma \triangleright f)$  is uniquely determined by the requirement

$$\begin{array}{c} \text{---} y \text{---} c \\ | \\ \boxed{\phantom{0}} \alpha(g) \\ | \\ x \text{---} {}_e\langle n, m \rangle \\ | \\ \boxed{\phantom{0}} \beta(\gamma \triangleright f) \\ | \\ \text{---} c \end{array} = \begin{array}{c} \text{---} y \text{---} c \text{---} m \\ | \\ \boxed{\phantom{0}} g \\ | \\ x \text{---} n \text{---} m \\ | \\ \gamma \boxed{\phantom{0}} f \\ | \\ y \text{---} c \end{array}, \quad (6.7)$$

for all  $g \in \mathbf{Hom}(y \triangleright (c \triangleright m), x \triangleright n)$ . From the  $\mathcal{C}$ -compatibility of  $\Theta$  and equation (6.2) one obtains that the second expression is given by

$$\begin{array}{c} \text{---} y \text{---} c \text{---} m \\ | \\ \boxed{\phantom{0}} g \\ | \\ x \text{---} n \text{---} m \\ | \\ \gamma \boxed{\phantom{0}} f \\ | \\ c \end{array} = \begin{array}{c} \text{---} y \text{---} c \\ | \\ \boxed{\phantom{0}} \alpha(g) \\ | \\ x \text{---} {}_e\langle n, m \rangle \\ | \\ \gamma \boxed{\phantom{0}} \beta(f) \\ | \\ c \end{array}. \quad (6.8)$$

The uniqueness result of Proposition 6.1 implies that  $\beta(\gamma \triangleright f) = \gamma \otimes \beta(f)$ .  $\square$

Consider a module category  ${}_e\mathcal{M}$  with module trace. We construct a coalgebra structure for  ${}_e\langle m, m \rangle$  for  $m \in \mathcal{M}$  in analogy to the construction of the algebra structure of

${}_e\langle m, m \rangle$  in Subsection 3.1. First we define the internal coevaluation  $\text{coev}_{n,m} : m \rightarrow {}_e\langle n, m \rangle \triangleright n$  as

$$\text{coev}_{n,m} = \beta^{-1}(\text{id}_{{}_e\langle n, m \rangle}) \hat{=} \begin{array}{c} m \\ \diagup \quad \diagdown \\ \boxed{\phantom{m}} \\ \diagdown \quad \diagup \\ n \end{array} . \quad (6.9)$$

Hence  $\text{coev}_{n,m}$  is characterised uniquely by the property that for all  $f \in \text{Hom}({}_e\langle n, m \rangle \triangleright n, m)$ :

$$\begin{array}{c} {}_e\langle n, m \rangle \\ \boxed{\phantom{m}} \\ f \\ \boxed{\phantom{m}} \\ m \end{array} = \begin{array}{c} \text{O} \\ \boxed{\phantom{m}} \\ \alpha(f) \end{array} . \quad (6.10)$$

Next we define the internal comultiplication  $\Delta_{m,n,k} : {}_e\langle m, k \rangle \rightarrow {}_e\langle n, k \rangle \otimes {}_e\langle m, n \rangle$  by

$$\begin{array}{c} k \\ \diagup \quad \diagdown \\ m \\ \diagdown \quad \diagup \\ n \end{array} = \beta \left( \begin{array}{c} k \\ \diagup \quad \diagdown \\ n \\ \diagdown \quad \diagup \\ m \end{array} \right) , \quad (6.11)$$

and the internal counit  $\epsilon : {}_e\langle m, m \rangle \rightarrow 1$  as

$$\epsilon = \beta \left( \begin{array}{c} \text{I} \\ \boxed{\phantom{m}} \\ m \end{array} \right) \hat{=} \begin{array}{c} \text{I} \\ \boxed{\phantom{m}} \\ m \end{array} . \quad (6.12)$$

**Lemma 6.3** *For all morphisms  $f \in \text{Hom}({}_e\langle n, k \rangle \triangleright n, c \triangleright m)$ ,*

$$\begin{array}{c} k \\ \diagup \quad \diagdown \\ n \\ \boxed{\phantom{m}} \\ f \\ \boxed{\phantom{m}} \\ c \end{array} \xrightarrow{\beta} \begin{array}{c} k \\ \diagup \quad \diagdown \\ m \\ \boxed{\phantom{m}} \\ \beta(f) \\ \boxed{\phantom{m}} \\ c \end{array} . \quad (6.13)$$

*Proof:* The proof is analogous to the proof of Lemma 3.3.  $\square$

**Proposition 6.4** *Let  ${}_e\mathcal{M}$  be a module category with module trace. For any object  $m \in \mathcal{M}$ , the internal hom  ${}_e\langle m, m \rangle$  is canonically a coalgebra object.*

*Proof:* The proof is analogous to the proof of Proposition 3.5.  $\square$

It remains to prove one more compatibility condition of  $\alpha$  and  $\beta$  before we can show that  ${}_e\langle m, m \rangle$  is a Frobenius algebra.

**Lemma 6.5** *Consider the morphism  $\text{coev}_{n,k} \circ \text{ev}_{l,k} : {}_e\langle l, k \rangle \triangleright l \rightarrow {}_e\langle n, k \rangle \triangleright n$ . By applying  $\alpha$  and  $\beta$  to this morphism we obtain the internal comultiplication and internal*



multiplication, respectively. In graphical terms:

$$\alpha \left( \begin{array}{c} \text{diagram with } l \text{ and } n \text{ strands} \end{array} \right) = \begin{array}{c} \text{diagram with } k, l, n \text{ strands} \end{array}, \quad (6.14)$$

and

$$\beta \left( \begin{array}{c} \text{diagram with } l \text{ and } n \text{ strands} \end{array} \right) = \begin{array}{c} \text{diagram with } k, l, n \text{ strands} \end{array}. \quad (6.15)$$

*Proof:* Define  $\Psi = \text{coev}_{n,k} \circ \text{ev}_{l,k}$ . First we compute  $\beta(\Psi)$  using equation (6.2). For all  $f \in \text{Hom}({}_e\langle n, k \rangle) \triangleright n, {}_e\langle l, k \rangle \triangleright l$ ,

$$\begin{array}{c} \text{diagram with } k, n \text{ strands} \end{array} \alpha(f) = \begin{array}{c} \text{diagram with } k, n \text{ strands} \end{array} f = \begin{array}{c} \text{diagram with } k, n \text{ strands} \end{array} \alpha(f), \quad (6.16)$$

where the last step involved equation (6.10) and Lemma 3.3. This proves that  $\beta(\Psi)$  is equal to the internal multiplication.

Next we calculate for all  $g \in \text{Hom}({}_e\langle n, k \rangle \triangleright n, {}_e\langle l, k \rangle \triangleright l)$ ,

$$\begin{array}{c} \text{diagram with } k, l \text{ strands} \end{array} \alpha(\Psi) = \begin{array}{c} \text{diagram with } k, l \text{ strands} \end{array} g = \begin{array}{c} \text{diagram with } k, l \text{ strands} \end{array} \beta(g), \quad (6.17)$$

where in the last step we used the definition of  $\text{ev}_{l,k}$  and Lemma 6.3. Since the trace on  $\mathcal{C}$  is non-degenerate and  $\beta$  an isomorphism, we conclude that  $\alpha(\Psi)$  is equal to the internal multiplication.  $\square$

**Theorem 6.6** *Let  $\mathcal{C}$  be a pivotal category and let  $\mathcal{M}$  be a  $\mathcal{C}$ -module category with module trace. For all non-zero  $m \in \mathcal{M}$ ,  ${}_e\langle m, m \rangle$  is a Frobenius algebra in  $\mathcal{C}$ . If  $m$  is a simple object then  ${}_e\langle m, m \rangle$  is a special haploid symmetric Frobenius algebra with  $\dim^{\mathcal{C}}({}_e\langle m, m \rangle) > 0$ .*

*Proof:* We show that the relations from Definition 2.5 are satisfied. Define the following

morphisms for  $k, l, n, r \in \mathcal{M}$ :

$$f_1 = \text{diagram} , \quad f_2 = \text{diagram} , \quad (6.18)$$

$$f_3 = \text{diagram} , \quad f_4 = \text{diagram} . \quad (6.19)$$

Lemma 6.5, the compatibility of  $\beta$  and the module action according to Proposition 6.2 and the associativity of the internal multiplication together imply

$$\beta(f_1) = \beta(f_3), \text{ hence } f_1 = f_3. \quad (6.20)$$

Similarly, as a consequence of Lemma 3.2, the coassociativity of the internal comultiplication and Lemma 6.5, we obtain

$$\alpha(f_2) = \alpha(f_4), \text{ hence } f_2 = f_4. \quad (6.21)$$

It follows that  $\alpha(f_1) = \alpha(f_3)$ , or in graphical terms

$$\text{diagram} = \text{diagram} , \quad (6.22)$$

where we again used compatibility of  $\alpha$  and the module structure as well as Lemma 6.5. Similarly we conclude that  $\beta(f_2) = \beta(f_4)$ . Together with Lemma 6.5 and Proposition 6.2 this implies

$$\text{diagram} = \text{diagram} . \quad (6.23)$$

If we restrict attention to the case where all objects are equal to  $m$ , we see that  ${}_e\langle m, m \rangle$  satisfies the relations (2.12) defining a Frobenius algebra. Let now  $m \in \mathcal{M}$  be simple. Then the identity  $\text{Hom}(1, {}_e\langle m, m \rangle) \simeq \text{Hom}(m, m) \simeq \mathbb{C}$  implies that  ${}_e\langle m, m \rangle$  is haploid. Recall that  $\eta_m$  and  $\epsilon_m$  denote the internal unit and counit, respectively. Equation (6.2)

shows that  $\epsilon_m \circ \eta_m = \Theta_m(\text{id}_m) \neq 0$ . Also by the symmetry of  $\Theta$  and by equation (6.10),

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \end{array} = \dim^{\mathcal{C}}({}_e\langle n, m \rangle). \quad (6.24)$$

As  $m$  is simple, this implies

$$\text{ev}_{n,m} \circ \text{coev}_{n,m} = \frac{\dim^{\mathcal{C}}({}_e\langle n, m \rangle)}{\dim^{\Theta}(m)} \cdot \text{id}_m. \quad (6.25)$$

Furthermore, combining Lemma 6.5 and Lemma 3.3, we obtain

$$\alpha \left( \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) = \begin{array}{c} \text{Diagram 3} \end{array}. \quad (6.26)$$

Together with equation (6.24) this implies

$$\mu_{m,n,m} \circ \Delta_{m,n,m} = \frac{\dim^{\mathcal{C}}({}_e\langle n, m \rangle)}{\dim^{\Theta}(m)} \cdot \text{id}_{\mathcal{C}\langle m, m \rangle}. \quad (6.27)$$

By setting  $m = n$  we find that  ${}_e\langle m, m \rangle$  is a special haploid Frobenius algebra, since by Proposition 5.7,  $\dim^{\mathcal{C}}({}_e\langle m, m \rangle) > 0$ . Due to Lemma 2.9,  ${}_e\langle m, m \rangle$  is also a symmetric algebra.  $\square$

We will now prove the converse of Theorem 6.6. For this we require the following result.

**Lemma 6.7** *Let  $A$  be a normalised special haploid Frobenius algebra in a pivotal fusion category  $\mathcal{C}$ . Then  $\dim^{\mathcal{C}}(M) \neq 0$  for all simple modules  $M \in \text{Mod}_{\mathcal{C}}(A)$ .*

*Proof:* The proof is an modification of the proof that all dimensions of simple objects in a pivotal fusion category are non-zero, see [1, Lemma 2.4.1]. We use the pivotal structure to identify left and right dual objects. First note that by Lemma 2.9,  $A$  is symmetric and for a symmetric Frobenius algebra,

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array}. \quad (6.28)$$

This follows from the fact that the left hand side is the inverse of the morphism on the left of equation (2.14), while the right hand side is the inverse morphism of the right hand side of (2.14), hence both have to agree.

Let  $(M, \rho)$  be a simple  $A$ -module. Proposition 2.10 implies  $\mathcal{C} = \mathbf{Hom}_A(M, M) \simeq \mathbf{Hom}_{\mathcal{C}}(1, M \otimes_A M^*)$ . It is sufficient to show that there are non-zero maps  $\text{coev}_M^A : 1 \rightarrow M \otimes_A M^*$  and  $\text{ev}_M^A : M \otimes_A M^* \rightarrow 1$  for which the diagram

$$\begin{array}{ccc} & & 1 \\ & \swarrow \text{coev}_M & \downarrow \text{coev}_M^A \\ M \otimes M^* & \xrightarrow{P} & M \otimes_A M^* \\ & \searrow \text{ev}_M & \downarrow \text{ev}_M^A \\ & & 1 \end{array} \quad (6.29)$$

commutes. The semisimplicity of  $\mathcal{C}$  then implies that the composition  $\text{ev}_M \circ \text{coev}_M$  is non-zero. We obtain  $\text{ev}_M^A$  from the universal property of the cokernel by observing that  $\text{ev}_M \circ (\rho \otimes \text{id}_{M^*}) = \text{ev}_M \circ (\text{id}_M \otimes \rho_{M^*})$  as morphisms  $M \otimes A \otimes M^* \rightarrow 1$ . Here  $\rho_{M^*}$  is defined by (2.15). For  $\text{coev}_M^A$  we have to show that  $P \circ \text{coev}_M \neq 0$ , where  $P$  is the projector (2.16). We compute

$$P \circ \text{coev}_M \hat{=} M \quad M^* = M \quad M^* = \quad (6.30)$$

$$= M \quad M^* \stackrel{(6.28)}{=} M \quad M^* = M \quad M^* = \quad (6.31)$$

$$\stackrel{(2.13)}{=} M \quad M^* = M \quad M^* = \text{coev}_M. \quad (6.32)$$

This proves the statement.  $\square$

**Proposition 6.8** *Let  $A$  be a special haploid Frobenius algebra in  $\mathcal{C}$ . Then the  $\mathcal{C}$ -module category of right  $A$ -modules,  $\mathbf{Mod}_{\mathcal{C}}(A)$  has a module trace induced by the trace on  $\mathcal{C}$ . In particular,  $A$  satisfies  $\dim^{\mathcal{C}}(A) > 0$ .*

*Proof:* The symmetry and  $\mathcal{C}$ -compatibility follow from the properties of the trace  $\text{tr}^{\mathcal{C}}$  in  $\mathcal{C}$ . We only have to show that the induced pairing on the  $\mathbf{Hom}$ -spaces of  $\mathbf{Mod}_{\mathcal{C}}(A)$  is non-degenerate. According to Lemma 5.1 it is sufficient to show that all simple modules  $m$  over  $A$  have  $\dim^{\mathcal{C}}(m) \neq 0$ . This follows from Lemma 6.7.

Consider the quantum dimension of  $A$ . Since  $A$  is haploid it is a simple module over itself. The inner hom object of  $\mathbf{Mod}_{\mathcal{C}}(A)$  is given by the tensor product over  $A$ , hence  ${}_e\langle A, A \rangle = A \otimes_A A = A$ , see e.g. [10] for the last equality. The statement now follows from Proposition 5.7.  $\square$

We have established the correspondence between module traces and Frobenius algebras. If  ${}_e\mathcal{M}$  is a module category with module trace, the dimensions of simple objects in general

change under the equivalence  ${}_e\mathcal{M} \ni n \mapsto {}_e\langle m, n \rangle \in \mathbf{Mod}_e({}_e\langle m, m \rangle)$  with  $m \in \mathcal{M}$  a simple object. The following lemma allows one to calculate the relevant scaling factor.

**Lemma 6.9** *Let  ${}_e\mathcal{M}$  be a module category with module trace. Let  $m, n \in \mathcal{M}$  be simple objects. Then*

$$\dim^e({}_e\langle m, n \rangle) = \frac{\dim^e({}_e\langle m, m \rangle)}{\dim^\Theta(m)} \cdot \dim^\Theta(n). \quad (6.33)$$

Under the equivalence  ${}_e\mathcal{M} \simeq \mathbf{Mod}_e({}_e\langle m, m \rangle)$  the dimensions of simple objects are scaled by  $\frac{\dim^e({}_e\langle m, m \rangle)}{\dim^\Theta(m)}$ .

*Proof:* Set  $d_i = \dim^\Theta(m_i)$ . From Proposition 5.7, we obtain

$$Q_{ij} = \frac{d_i \overline{d_j} \dim(\mathcal{C})}{\sum_k |d_k|^2}. \quad (6.34)$$

This implies

$$\begin{aligned} \dim^e({}_e\langle m_j, m_i \rangle) &= d_i \cdot \frac{|d_j|^2 \dim(\mathcal{C})}{d_j \sum_k |d_k|^2} \\ &= d_i \cdot \frac{\dim^e({}_e\langle m_j, m_j \rangle)}{d_j}, \end{aligned} \quad (6.35)$$

where we again used equation (6.34) in the last step. Setting  $m = m_j$  and  $n = m_i$  then proves the claim.  $\square$

Finally we interpret our result using the notion of Morita equivalence of algebras (see [17]). Two algebras  $A, B \in \mathcal{C}$  are called Morita equivalent if the categories  $\mathbf{Mod}_e(A)$  and  $\mathbf{Mod}_e(B)$  are equivalent as module categories.

**Theorem 6.10** *Every separable indecomposable algebra  $A$  in a fusion category with a flexible pivotal structure is Morita equivalent to a special haploid symmetric Frobenius algebra.*

*Proof:* By definition of a flexible pivotal structure, the module category  $\mathbf{Mod}_e(A)$  possesses a module trace. By Theorem 6.6, this module category is equivalent to the module category corresponding to a special haploid Frobenius algebra.  $\square$

Together with Proposition 5.8 this implies the following:

**Corollary 6.11** *If an indecomposable algebra  $A$  in a pseudo-unitary fusion category  $\mathcal{C}$  is separable, then it is Morita equivalent to a special haploid symmetric Frobenius algebra.*

## A Graphical Calculus for Tensor Categories

We summarise the graphical calculus for tensor categories, see e.g. [1]. The symbol  $\hat{=}$  is used to indicate that a certain diagrammatic expression represents an algebraic expression. Objects in  $\mathcal{C}$  and the tensor product are represented by the following diagrams.

$$c \hat{=} \left| \begin{array}{c} | \\ c \end{array} \right|, \quad c \otimes d \hat{=} \left| \begin{array}{c} | \\ c \otimes d \end{array} \right| \quad \left| \begin{array}{c} | \\ d \end{array} \right| \quad \left| \begin{array}{c} | \\ c \end{array} \right|. \quad (A.1)$$

Morphisms are represented by labelled boxes, and we do not distinguish objects from their unit morphisms. All diagrams are read from top to bottom. The composition is given by vertical connection of boxes.

$$f : c \rightarrow d \hat{=} \begin{array}{c} | \\ c \\ \boxed{\phantom{f}} \\ | \\ d \end{array} f, \quad g \circ f \hat{=} \begin{array}{c} | \\ c \\ \boxed{\phantom{f}} \\ | \\ d \\ \boxed{\phantom{g}} \\ | \\ b \end{array} f = \begin{array}{c} | \\ c \\ \boxed{\phantom{g \circ f}} \\ | \\ b \end{array} g \circ f. \quad (\text{A.2})$$

An empty box represents a **Hom**-vector space:

$$\text{Hom}(c, d) \hat{=} \begin{array}{c} | \\ c \\ \boxed{\phantom{\text{Hom}(c, d)}} \\ | \\ d \end{array}. \quad (\text{A.3})$$

The tensor product of two morphisms  $f : c \rightarrow d$  and  $g : a \rightarrow b$  is depicted as follows:

$$f \otimes g \hat{=} \begin{array}{c} | \\ c \\ \boxed{\phantom{f}} \\ | \\ d \end{array} f \quad \begin{array}{c} | \\ a \\ \boxed{\phantom{g}} \\ | \\ b \end{array} g. \quad (\text{A.4})$$

The interchange law  $f \otimes g = (f \otimes \text{id}_a)(\text{id}_d \otimes g) = (\text{id}_c \otimes g)(f \otimes \text{id}_b)$  has the following graphical expression:

$$\begin{array}{c} | \\ c \\ \boxed{\phantom{f}} \\ | \\ d \end{array} f \quad \begin{array}{c} | \\ a \\ \boxed{\phantom{g}} \\ | \\ b \end{array} g = \begin{array}{c} | \\ c \\ \boxed{\phantom{f}} \\ | \\ d \end{array} f \quad \begin{array}{c} | \\ a \\ \boxed{\phantom{g}} \\ | \\ b \end{array} g = \begin{array}{c} | \\ c \\ \boxed{\phantom{f}} \\ | \\ d \end{array} f \quad \begin{array}{c} | \\ a \\ \boxed{\phantom{g}} \\ | \\ b \end{array} g. \quad (\text{A.5})$$

The graphical notation suppresses the unit object and the associativity constraint in  $\mathcal{C}$ . Due to Mac Lane's coherence theorem for monoidal categories, a graphical expression uniquely defines a morphisms in  $\mathcal{C}$  once parentheses and unit objects are specified for the incoming and outgoing objects. The evaluation and coevaluation morphisms for the right duals are depicted as follows:

$$\text{ev}_c \hat{=} c^* \cup c, \quad \text{coev}_c \hat{=} c \cap c^*, \quad (\text{A.6})$$

and the rigidity axioms read:

$$\begin{array}{c} \cup \\ c \quad c^* \end{array} c = c, \quad \begin{array}{c} \cap \\ c^* \quad c \end{array} c^* = c^*. \quad (\text{A.7})$$

The graphical notation for left duals is analogous. If  $\mathcal{C}$  is a pivotal category, we will often suppress the pivotal isomorphism and identify right duals and left duals, so the left

evaluation reads

$$c \cup c^* \cong c \cup \begin{array}{c} c^* \\ \boxed{\phantom{a_c}} \\ *c \end{array} a_c. \quad (\text{A.8})$$

The right dual of a morphism  $f : c \rightarrow d$  is defined by:

$$d^* \cup \begin{array}{c} c \\ \boxed{f} \\ d \end{array} \cup c^*. \quad (\text{A.9})$$

The left dual of a morphism is defined analogously using the left duality, and the map

$$\begin{array}{c} c \quad d \\ \boxed{\phantom{x}} \\ x \end{array} \mapsto \begin{array}{c} \phantom{c} \quad \phantom{d} \\ \cup \\ \begin{array}{c} c \quad d \\ \boxed{\phantom{x}} \\ *c \quad x \end{array} \end{array} \quad (\text{A.10})$$

yields an isomorphism  $\text{Hom}(c \otimes d, x) \rightarrow \text{Hom}(d, {}^*c \otimes x)$ .

## Acknowledgements

The author thanks Catherine Meusburger for many helpful discussions, support and detailed reading of the draft. He furthermore thanks Christoph Schweigert, Martin Mombelli and Ingo Runkel for their hospitality, useful discussions and support, Martina Lanini and Winston Fairbairn for proofreading and Torsten Schönfeld for his help with Latex and the implementation of the diagrams.

This work was funded by the research grant ME 3425/1-1 in the Emmy-Noether program of the German Research Foundation.

## References

- [1] B. Bakalov and A. Kirillov. *Lectures on tensor categories and modular functors*. University Lecture Series, AMS., 2001.
- [2] J. W. Barrett and B. W. Westbury. Invariants of piecewise-linear 3-manifolds. *Trans. Am. Math. Soc.*, 348(10):3997–4022, 1996.
- [3] B. Bartlett. *On unitary 2-representations of finite groups and topological quantum field theory*. PhD thesis, University of Sheffield, 2009, arXiv:0901.3975.
- [4] K. Costello. Topological conformal field theories and Calabi-Yau categories. *Adv. Math.*, 210(1):165–214, 2007.
- [5] A. Davydov, M. Mueger, D. Nikshych, and V. Ostrik. The Witt group of non-degenerate braided fusion categories. 2010, arXiv:1009.2117.
- [6] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. Tensor categories. <http://www-math.mit.edu/~etingof/tenscat.pdf>, 2009. Lecture notes, MIT open course.
- [7] P. Etingof, D. Nikshych, and V. Ostrik. On fusion categories. *Ann. of Math. (2)*, 162(2):581–642, 2005.

- [8] J. Froehlich, J. Fuchs, I. Runkel, and C. Schweigert. Correspondences of ribbon categories. *Adv. Math.*, 199(1):192–329, 2006.
- [9] J. Fuchs, I. Runkel, and C. Schweigert. TFT construction of RCFT correlators. I: Partition functions. *Nucl. Phys., B*, 646(3):353–497, 2002.
- [10] J. Fuchs and C. Schweigert. Category theory for conformal boundary conditions. In *Vertex operator algebras in mathematics and physics (Toronto, ON, 2000)*, volume 39 of *Fields Inst. Commun.*, pages 25–70. Amer. Math. Soc., 2003.
- [11] M. R. Gaberdiel and I. Runkel. Logarithmic bulk and boundary conformal field theory and the full centre construction. 2012, arXiv:1201.6273.
- [12] J. Greenough. *Bimodule Categories and monoidal 2-structures*. PhD thesis, University of New Hampshire, 2010.
- [13] J. Greenough. Monoidal 2-structure of bimodule categories. *J. Algebra*, 324(8):1818–1859, 2010.
- [14] A. Kapustin and N. Saulina. Surface operators in 3d Topological Field Theory and 2d Rational Conformal Field Theory. 2010, arXiv:1012.0911v1.
- [15] G. Moore and N. Seiberg. Classical and quantum conformal field theory. *Comm. Math. Phys.*, 123(2):177–254, 1989.
- [16] M. Mueger. From subfactors to categories and topology. I: Frobenius algebras in and Morita equivalence of tensor categories. *J. Pure Appl. Algebra*, 180(1-2):81–157, 2003.
- [17] V. Ostrik. Module categories, weak Hopf algebras and modular invariants. *Transform. Groups*, 8(2):177–206, 2003.
- [18] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103(3):547–597, 1991.
- [19] C. Schweigert, J. Fuchs, and I. Runkel. Categorification and correlation functions in conformal field theory. In *International Congress of Mathematicians. Vol. III*, pages 443–458. Eur. Math. Soc., Zürich, 2006.
- [20] V. G. Turaev. *Quantum invariants of knots and 3-manifolds. 2nd revised ed.* de Gruyter Studies in Mathematics, 2010.